

IRREGULAR HOMOTOPIES OF HARMONIC MAPS OF SURFACES

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Abstract of a talk given on 1 June 1995 at Universität Bonn.

The purpose of this talk is to discuss hypotheses that might be necessary to insure that an energy minimizing map $f : \mathcal{R} \rightarrow N^3, h$ of a Riemann surface into a Riemannian 3-manifold is immersed. The question is interesting geometrically as well as important technically, as a positive result would provide for some substantial technical convenience in the analysis of some 3-manifold geometry via harmonic maps. Unfortunately for this purpose, the results we will discuss will point to there being no satisfying hypotheses which would guarantee that a harmonic map is an immersion.

We begin with some background. Given a map $f : (M^2, g) \rightarrow (N^n, h)$ from a Riemannian surface to a Riemannian n -manifold, one can measure the total energy $E(f : M \rightarrow N) = \int_M \text{tr}_g f^* h dA(g)$. We recall that a critical point of this functional is called a harmonic map, and that evidently this functional is conformally invariant, so that a harmonic map depends only on the structure of M as a Riemann surface. We also recall the historical background that if (N, h) is a non-positively curved coconvex manifold, then every homotopy class of maps $[f_0 : M \rightarrow N]$ contains a unique-energy minimizing smooth representative (Eells-Sampson [ES], Al'ber [A], Hartman [H]).

For this reason, we will restrict ourselves to the initial hypothesis that (N, h) is non-positively curved.

There is historical precedent for our question. Indeed, Schoen-Yau [SY] and Sampson [Sa] independently proved that a degree one harmonic map between surfaces was immersed, and Osserman-Gulliver-Alt ([O], [G], [Al]) proved that the Douglas-Rado solution of the Plateau problem (recall that such a minimal map is both energy minimizing and conformal) is immersed. Perhaps more relevant for the considerations of this talk is the theorem of Freedman-Hass-Scott [FHS] which stated that an area-minimizing surface homotopic to the fiber $M^2 \times \{0\}$ in a quasi Fuchsian manifold $N \approx M^2 \times (-1, 1)$ is an embedding. (Recall that a quasi Fuchsian manifold is a quotient $\mathbf{H}^3/\Gamma$ where $\Gamma \subset \text{PSL}(2, \mathbf{C})$ is a quasi conformal deformation of a Fuchsian representation $\rho : \pi_1 M \rightarrow \text{PSL}(2, \mathbf{R})$).

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We first consider some compressible examples.

Example 1. Consider the harmonic maps Dirichlet problem of mapping a cylinder C into $\mathbf{E}^3$ taking the boundary circles onto two parallel circles C_1, C_2 on $\mathbf{E}^3$ with antipodal boundary values (i.e., parallel points $(\theta, 0)$ and $(\theta, 1)$ on $\partial C = \partial(S^1 \times [0, 1]) = \{0 \leq \theta < 2\pi\} \times \{0, 1\}$ get mapped to antipodal points $(\theta, 0)$ and $(\theta + \pi, 1)$ on C_1, C_2 in the natural notation. By symmetry, the resulting harmonic map collapses the central circle $S^1 \times \{\frac{1}{2}\}$ to a point.

Example 2. Let H be a handlebody of genus $g \geq 2$; we consider a harmonic representative of the class of the inclusion $\partial H \hookrightarrow H$. In particular, we endow H with a hyperbolic structure so that its convex hull is two-dimensional. Then the image of any harmonic map must be contained within this convex hull and hence have fold lines transverse to geodesics which are compressed.

It seems natural to consider then some incompressible examples.

Example 3. (Figure 1) The map $(u(x, y), v(x, y), w(x, y)) = (x, x^2 - y^2 + 2xy, y^3 - 3x^2y)$ maps a disc harmonically and injectively into $\mathbf{E}^3$; the map folds along the fold line $\{y = x\}$ where the differential has rank 1. We could regard this as solving a Dirichlet problem into a convex domain in $\mathbf{E}^3$ with unsmooth Dirichlet boundary values, but as an injection.

Example 4. (Figure 2) The map $(u(x, y), v(x, y), w(x, y)) = (x - x^3/3 + xy^2, y - y^3/3 + x^2y, xy)$ maps a disc of radius 2 into $\mathbf{E}^3$ with compact singular set the unit circle. Here the circle consists of fold points and four cusp points. It is possible to produce such examples with any even number of cusp points exceeding 2.

Example 5. Theorem: There is a quasi Fuchsian manifold $N \approx M^2 \times (-1, 1)$ and a Riemann surface $\mathcal{R} \times M^2$ such that the harmonic map $f : M^2 \rightarrow N$ homotopic to the map $M \rightarrow M \times \{0\}$ is not an immersion.

Remarks. By the results of Schoen-Yau [SY] and Sampson [Sa], such a target cannot be Fuchsian, or else the harmonic map would factor through the totally geodesic representative of $M^2 \times \{0\}$. Also, by the results of Freedman-Hass-Scott [FHS], the map cannot be area minimizing. Thus we look for a non-conformal map.

Sketch of Proof. We argue indirectly and assume that all such maps of all such surfaces into all such manifolds are immersions and derive a contradiction. We then consider, for a fixed Riemannian target N , the Teichmüller space $\mathcal{T}$ of domain Riemann surfaces; we know that there is one area minimizing map in this space, hence one embedding. We also know that all the maps parametrized by this space $\mathcal{T}$ are homotopic. The goal of the proof is to produce a harmonic map f^* which is in a different regular homotopy class, so that a homotopy between the area minimizing map and the map f^* could not be through immersions.

To do this we combine several ingredients. First, we recall the theorem of Hass and Hughes [HH] which classified the regular homotopy classes of immersions of M into $M \times (-1, 1)$ as being given by $H^1(M, \mathbf{Z}_2)$ and more geometrically, by their “kinks”: here we

say that we have added a kink along a curve γ to a surface M if we obtain the kinked surface by first removing a neighborhood N of the curve $\gamma \subset M$ and replacing it with $N^* = "a" \times S^1$, the product of the shape of the letter "a" with the circle.

Thus, the question becomes: how does one produce harmonic maps with control on their global topology. To do this we combine three ingredients. First we observe that the grafting construction of Maskit-Hejhal-Sullivan/Thurston ([M], [He], [ST]) to produce exotic projective structures $\mathcal{F}$ on the upper domain of discontinuity Ω^+ of $\mathbf{H}^3/\Gamma$ has the effect, by taking the associated convex hull of the structure $\mathcal{F}$, of producing pleated surfaces $\mathcal{P}$ inside $\mathbf{H}^3/\Gamma$ "with kinks". We then adapt the theory of Minsky [Mi] on high energy harmonic maps from surfaces to hyperbolic three-manifolds to produce a harmonic map with image lying near the image of $\mathcal{P}$, so that this image also has a kink. Finally, some estimates on high energy harmonic maps and a maximum principle argument rule at any spurious kinks.

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