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Ray structures on Teichmüller space

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Contents

1. Introduction

1.1. Overview

Teichmüller space admits several ray structures, each arrived at through a similar process. One begins with a pair of points in Teichmüller space and decides whether to construe those points as complex structures or hyperbolic structures on that surface. Then, one chooses a variational problem to solve, the solution to which defines an auxiliary object on the surface, for example a holomorphic differential or a geodesic lamination and a scaling constant. A family of pairs where the auxiliary object is fixed projectively then defines a path in Teichmüller space.

Probably the most famous example of this process are the Teichmüller geodesics. Here, we attempt to minimize the quasiconformal dilatation between a pair of Riemann surfaces, finding a solution that may be described in terms of holomorphic quadratic differentials on each surface; looking for families in which the projective class of such a differential on one of the surfaces is fixed defines a Teichmüller ray.

Thurston [90] developed an analogue of this perspective in the 1980's where he viewed points of Teichmüller space as hyperbolic structures on a surface. Then, one sought minimizers of the Lipschitz stretch of maps between the surfaces, obtained by a map with maximal stretch locus a chain-recurrent geodesic lamination. When, for example, that lamination was maximal, one could consider the family of target surfaces for which the lamination was fixed, and this family defined a Thurston geodesic for his (asymmetric) metric.

There are several other families of minimizers, for example Kerckhoff's "lines of minima" [50], but here we focus on ones related to minimization of L^2 Energy. Because of a conformal invariance in two dimensions, this energy depends on the hyperbolic structure of the target but only the conformal (hence complex) structure of the domain, and so occupies a middle ground between the two variational problems we opened with. Here, also the solution, a harmonic map, determines and may be understood through the use of a holomorphic quadratic (Hopf) differential, and once again the families of targets who projectively share such a differential foliate Teichmüller space.

As a summary comment, though these ray structures are developed through similar processes, the Teichmüller geodesic rays, the Thurston geodesic rays and the harmonic map Hopf differential rays seem unrelated.

We need one more construction. A somewhat more obscure ray structure comes from seeking a family of Riemann surfaces whose Hopf differentials share one particular (horizontal) projective measured foliation. (See [88].) We can imagine these as defining a type of "dual" harmonic map rays which we will clarify a bit later.

With all this context in place, we may explain our goals in this paper. We show a collection of results that together portray the harmonic map rays structures as providing a transition between the Teichmüller ray structures and the Thurston ray structures. In particular, we show the following, stated roughly in this overview, and in terms of an initial harmonic map $u: X \to Y$ and of course in terms of the Hopf differential $\mathrm{Hopf}(X, Y)$.

First, if we allow the domain X to degenerate along harmonic map dual rays, then the harmonic map rays through Y converge to Thurston geodesics. (See Theorems 1.1 and 1.3.)

Second, if we allow the target Y to degenerate along harmonic map rays, the harmonic map dual rays through X converge to Teichmüller geodesics. (See Theorem 1.4.)

Third, there are natural ways that the Teichmüller and Thurston rays may be arranged into disks, either as Teichmüller disks in the complex case or into stretch-earthquake disks in the hyperbolic case. Of course, the Hopf differentials also admit a $\mathbb{C}^*$ action, and the results above on harmonic map rays defining a transition between Teichmüller and Thurston geodesics extend to the disk setting.

Finally, we take up some applications of these results. We were slightly careful in our description of the Thurston Lipschitz problem to distinguish solutions where the maximally stretched lamination was maximal (so that the complementary regions were ideal triangles) from the more general solutions. Thurston noted that in the non-maximal case, there was no canonical (Thurston) geodesic between the pair of surfaces. There have been a number of proposals for some more canonical geodesics between some pairs of surfaces in some settings (see [73], [41], [11]). Here, we observe that if we additionally require the geodesic from Y to Z to solve an energy-minimization problem in the sense of being a limit of harmonic map rays that proceed from Y through Z, then this results in a uniquely defined geodesic, a "harmonic stretch line". (See Theorem 1.6.)

Moreover, a simple extension of that technique produces a well-defined Thurston geodesic proceeding from a hyperbolic surface to a point on the Thurston boundary of Teichmüller space, represented by a projective measured foliation. Thus in Theorem 1.10 we show that the harmonic stretch rays from a point Y foliate Teichmüller space and accumulate only at their endpoint on the Thurston boundary: thus they provide an "exponential" map for the Thurston metric with visual boundary the Thurston boundary.

This "exponential" map for the Thurston metric defines one of two distinct versions we may define for the Thurston geodesic flow on the bundle over the Teichmüller space with fibers the space $\mathcal{PMF}$ of projective measured foliations: here the orbit of a hyperbolic surface X and a projective measured foliation $[\eta]$ is the harmonic stretch line which passes through X and converges to $[\eta]$ in the Thurston boundary in the forward direction. The other version of the Thurston geodesic flow is defined such that the orbit of $(X, [\eta])$ is the harmonic stretch line arising as the limit of harmonic map rays through X when the domain degenerates along the harmonic map dual ray determined by X and η. The first version of the Thurston geodesic flow describes the contracting foliation along its orbits (which is also the endpoint on the Thurston boundary) while the second version describes the stretching lamination. Both of these two flows commute with the mapping class group action, and hence descend to the bundle over moduli space with fiber $\mathcal{PMF}$.

Along the way, we show some other results, for example an extension (Theorem 14.1) of this theory to surfaces with geodesic boundary, proving that the optimal Lipschitz constants between hyperbolic surfaces with boundaries are always realized by some surjective

Lipschitz homeomorphisms. This verifies a conjecture of Alessandrini and Disarlo [3] (in the case of unpunctured surfaces).

In several places, we prove not only subconvergence of the approximating rays but actual convergence of the families. The technique here often comes down to regarding a harmonic map as an equivariant minimal graph over a hyperbolic surface with values in an $\mathbb{R}$-tree, and we prove a uniqueness theorem in that case. (See Theorem 5.7.) The difficulties here are both that the tree typically does not admit an orientation and the graphs we imagine have infinite diameter, but our result might still be regarded as in the spirit of the Jenkins–Serrin uniqueness theorem (see [44]). Having broached this topic, in the appendix we provide some completeness to the discussion by also proving a corresponding existence theorem for graphs with values in a real tree and asymptotic boundary values over hyperbolic domains with geodesic boundary (that result will also play a role in the proof of Lemma 7.4). (Here, by extending the range to trees, we extend the possible domains to surfaces whose polygonal ends are not limited to an even number of sides, as in [44] and elsewhere.)

In the next section, we define our terms and state our results somewhat more carefully.

1.2. Harmonic map rays and harmonic stretch rays

The remainder of this introductory section is devoted to stating our results. We quickly declare some notation: a fuller treatment of the constructions behind those definitions is given in §2 and §3.

1.2.1. Four old families of rays and one new one

Let S be an orientable closed surface of genus at least 2 and $\mathcal{T}(S)$ be the Teichmüller space of S (see §2.1 for the definition of $\mathcal{T}(S)$). A Teichmüller ray $\mathbf{TR}_{X,\Phi}\colon[1,\infty)\to\mathcal{T}(S)$ is defined by a base Riemann surface X and a quadratic differential Φ which is holomorphic on X (cf. §2.4). It is convenient to also denote that ray by $\mathbf{TR}_{X,\lambda}\colon[1,\infty)\to\mathcal{T}(S)$, where λ is the horizontal measured (or projective measured) lamination of Φ.

A Thurston stretch ray $\mathbf{SR}_{Y,\lambda}\colon[1,\infty)\to\mathcal{T}(S)$ is defined with base point a hyperbolic surface Y and a maximal geodesic lamination λ (cf. §2.6).

A harmonic map ray $\mathbf{HR}_{X,\Phi}\colon[1,\infty)\to\mathcal{T}(S)$ is the family of surfaces so that the harmonic map from X to any member of that family has Hopf differential proportional to Φ, a holomorphic quadratic differential on X (cf. Definition 3.1). We will also have need of the notation $\mathbf{HR}_{X,Y}\colon[1,\infty)\to\mathcal{T}(S)$ which indicates the restriction of that harmonic

Names of rays	Notation	Reference
Teichmüller ray	$\mathbf{TR}_{X,\Phi}$, $\mathbf{TR}_{X,\lambda}$, or $\mathbf{TR}_{X,Y}$	§2.4
Thurston stretch ray or line	$\mathbf{SR}_{X,\lambda}$ or $\mathbf{SL}_{X,\lambda}$	§2.6
Harmonic map ray	$\mathbf{HR}_{X,\Phi}$, $\mathbf{HR}_{X,\lambda}$, or $\mathbf{HR}_{X,Y}$	Definition 3.1
Harmonic map dual ray	$\mathbf{hr}_{Y,\lambda}$ or $\mathbf{hr}_{X,Y}$	Definition 3.2
Harmonic stretch ray or line	$\mathbf{HSR}_{X,Y}$ or $\mathbf{HSL}_{X,Y}$	Definition 12.1
Piecewise harmonic stretch ray or line	$\mathbf{PSR}_{Y,\lambda,f}$ or $\mathrm{PSL}_{Y,\lambda,f}$	Theorem 1.5

Table 1. Various rays and lines in this paper.

map ray to begin at the hyperbolic surface Y.

The most obscure previously defined ray we will consider initially is the harmonic map dual ray $\mathbf{hr}_{Y,\lambda}=\mathbf{hr}_{Y,\lambda}(t)$, defined by the condition that the horizontal measured foliation of the Hopf differential $\mathrm{Hopf}(\mathbf{hr}_{Y,\lambda}(t)\to Y)$ is measure equivalent to the measured lamination $t\lambda$ (cf. Definition 3.2). Alternatively, $\mathbf{hr}_{Y,\lambda}(t)$ is the conformal structure underlying the projection of the minimal surface which is a graph over $\widetilde{Y}$ in the product $\widetilde{Y}\times T_{t\lambda}$, where $T_{t\lambda}$ is the tree dual to the lift of the lamination $t\lambda$.

Eventually we will define a family of "harmonic stretch lines" (see Definition 12.1), a distinguished family of Thurston geodesics arising as limits of harmonic map rays.

For convenience, we collect all types of rays and lines used in this paper in Table 1, with names, notations, and references.

1.2.2. Foundational convergence results

We now state our first results on asymptotic relationships between these ray families.

THEOREM 1.1. *Let $Y\in\mathcal{T}(S)$ be a hyperbolic surface and λ a measured lamination. Then, the harmonic map rays $\mathbf{HR}_{X_t,Y}\colon[1,\infty)\to\mathcal{T}(S)$ converge to a Thurston geodesic locally uniformly as X_t diverges along the harmonic map dual ray $\mathbf{hr}_{Y,\lambda}\colon[1,\infty)\to\infty$.*

Moreover, if λ is maximal, then the limit Thurston geodesic is exactly the Thurston stretch ray $\mathbf{SR}_{Y,\lambda}\colon[1,\infty)\to\mathcal{T}(S)$ defined by Y and λ.

Remark 1.2. It is natural to ask whether the result still holds if X_t degenerates along the Teichmüller ray $\mathbf{HR}_{X_t,Y}\colon[1,\infty)\to\mathcal{T}(S)$. The answer is affirmative if λ is maximal (see Theorem 8.8). (We believe that answer holds in general, and we hope to take up this issue in a future paper.)

We also have the following compactness result.

Theorem 1.3. *For any fixed* $Y\in\mathcal{T}(S)$*, let* $X_n\in\mathcal{T}(S)$ *be any divergent sequence. Then, the sequence of harmonic map rays* $\mathbf{HR}_{X_n,Y}\colon[1,\infty)\to\mathcal{T}(S)$ *contains a subsequence which converges to some Thurston geodesic locally uniformly.*

The limit Thurston geodesics in Theorem 1.1 for non-maximal measured lamination will be characterized later as instances of a special class — called the "harmonic stretch rays" (Definition 12.1) — of "piecewise harmonic stretch rays" (see Theorem 1.5, §8).

1.2.3. Harmonic map dual rays and Teichmüller rays

Informally, these results we just presented assert that if we look at a family of harmonic map rays $\mathbf{HR}_{X_s,Y}$ through a hyperbolic structure Y and then degenerate the domains X_s appropriately, say along a harmonic map dual ray, then the rays limit on a Thurston geodesic.

It is natural to ask what happens to the limits of harmonic map rays if we degenerate in the other direction, say by letting the target surface, now Y_s, tend to infinity? We should expect some object to emerge that is defined in terms of the complex structure X. Roughly, the answer is that the relevant rays converge to Teichmüller geodesics through X. More formally, the rays through X to focus on are the "harmonic map dual rays" defined in §1.2.1. We then show that, if we degenerate appropriately, this time letting the target Y_s degenerate along a harmonic map ray, then the harmonic map dual rays through X increasingly approximate Teichmüller geodesics.

Precisely, fix a complex structure $X\in\mathcal{T}(S)$. Let Φ be a holomorphic quadratic differential on X. Let $Y_s:=\mathbf{HR}_{X,\Phi}(s)$; we will be degenerating the harmonic map dual rays defined by X and Y_s.

To that end, let $\lambda\in\mathcal{ML}(S)$ be the measured lamination which is measure equivalent to the horizontal foliation of Φ. This data defines a family of harmonic map dual rays through X as follows.

We say that a *harmonic map dual ray* $\mathbf{hr}_{Y,\lambda}$ is the set of points X_t such that

$$\mathrm{Hor}(\mathrm{Hopf}(X_t\to Y))=t\lambda$$

(cf. [88]). An alternative, perhaps more geometric, description ([100]) is that the universal cover $\widetilde{X}_t$ is the minimal surface graph over the universal cover $\widetilde{Y}$ in the product $\widetilde{Y}\times tT_\lambda$, where T_λ is the real tree dual to λ (here of course lifting equivariantly to universal covers).

Extending this definition to encompass a family of such harmonic map dual rays, indexed by s, we set $X_{s,t}\in\mathbf{hr}_{Y_s,\sqrt{s}\lambda}$ to be such that $\mathrm{Hor}(\mathrm{Hopf}(X_{s,t}\to Y_s))=t\sqrt{s}\lambda$. Then,

$X_{s,1} \equiv X$ for all $s>0$. (Again, the more geometric description is that $\widetilde{X}_{s,t}$ is the minimal surface graph over $\widetilde{Y}_s$ in the product $\widetilde{Y}_s \times t\sqrt{s}\, T_\lambda$, where T_λ is the real tree dual to λ.)

Then, in that setting, we prove our convergence result.

THEOREM 1.4. *The family of harmonic map dual rays*

$$\mathbf{hr}_{Y_s,\sqrt{s}\lambda}\colon [1,\infty) \longrightarrow \mathcal{T}(S)$$

converges, as $s \to +\infty$, *locally uniformly to the Teichmüller geodesic ray*

$$\mathbf{TR}_{X,\Phi}\colon [1,\infty) \longrightarrow \mathcal{T}(S).$$

1.3. Piecewise harmonic stretch lines and harmonic stretch lines

Theorems 1.1 and 1.3 provide for a collection of distinguished Thurston geodesics, those that arise as limits of harmonic map rays. We turn next to describing the locations of these special geodesics in Teichmüller space (and its Thurston compactification), finding existence and uniqueness theorems for such geodesics between given points in those spaces.

To begin, Theorem 1.1 enables us to generalize Thurston's construction of stretch lines (maps) from maximal laminations to non-maximal ones using harmonic maps.

A basic tool for us will be "piecewise harmonic stretch maps". These maps will have the same structure of the limits of the harmonic maps defined by a harmonic map ray — and indeed they will often arise as such limits — but they are not required a priori to have any compatibility between components.

Later, we will combine their properties with a uniqueness result (cf. Theorem 1.13 below) to argue for Theorem 1.1, with the convergence property in that theorem relying on a compatibility across the components inherited from the limiting process.

THEOREM 1.5. (Piecewise harmonic stretch map) *Let* $Y \in \mathcal{T}(S)$ *be any closed hyperbolic surface, and let* λ *be any geodesic lamination. Then, for any harmonic diffeomorphism* $f\colon X \to Y\setminus\lambda$ *from some (possibly disconnected) punctured surface* X, *there is a new hyperbolic surface*

$$Y_t := \mathrm{PSL}_{Y,\lambda,f}(t) \in \mathcal{T}(S)$$

depending analytically on $\{t>0\}$ *such that the following conditions hold:*

(a) *the induced map* $f_t\colon X \to Y_t\setminus\lambda$ *is a harmonic diffeomorphism with Hopf differential* $t\mathrm{Hopf}(f)$;

(b) *for any* $0<s<t$, *the map* $f_t \circ f_s^{-1}$ *extends to a homeomorphism from* Y_s *to* Y_t *that is* $\sqrt{t/s}$*-Lipschitz with (pointwise) Lipschitz constant strictly less than* $\sqrt{t/s}$ *in* $Y_s\setminus\lambda$, *but exactly expands arc length on* λ *by the constant factor* $\sqrt{t/s}$.

The family of hyperbolic structures $\mathrm{PSL}_{Y,\lambda,f}(t)$ constructed above is called a *piecewise harmonic stretch line*. It admits a canonical orientation coming from the orientation of the positive real ray $\{t>0\}$. In that orientation, a piecewise harmonic stretch line is a (reparameterized) geodesic in the Thurston metric. Whenever we say a piecewise harmonic stretch line, we mean a directed line. Similarly to Thurston's construction of concatenation of stretch lines, one can construct a geodesic between any two points in Teichmüller space by a concatenation of piecewise harmonic stretch line segments.

Now, in addition to the piecewise harmonic stretch lines described in Theorem 1.5 above, an important focus for us will be families in a subclass we call *harmonic stretch lines*. A piecewise harmonic stretch line is called a *harmonic stretch line* if it is the limit of a sequence of harmonic map rays. Given hyperbolic surfaces $Y, Z \in \mathcal{T}(S)$, a harmonic stretch line passing through Y to Z will be a limit of a family of harmonic map rays from base points $X_n \in \mathcal{T}(S)$ that all proceed through Y to Z, as the base points X_n degenerate in $\mathcal{T}(S)$. Of course, as in the case of the piecewise harmonic stretch lines, these harmonic stretch lines are also directed.

Our basic theorem on these harmonic stretch lines is the following.

THEOREM 1.6. (Uniqueness of harmonic stretch lines) *For any two distinct hyperbolic surfaces $Y, Z \in \mathcal{T}(S)$, there exists a unique harmonic stretch line proceeding from Y through Z.*

Remark 1.7. From the construction of Thurston stretch lines and piecewise harmonic stretch lines, we see that every Thurston stretch line is a piecewise harmonic stretch line (see Lemma 7.10). However, Thurston stretch lines are not necessarily harmonic stretch lines. In fact, a Thurston stretch line is a harmonic stretch line if and only if the defining maximal lamination is *chain-recurrent*; see Corollary A.6.

Remark 1.8. The constructions above extend to the case when Y is a hyperbolic surface with geodesic boundary components. Using the uniqueness result Theorem 1.6, we verify a conjecture (cf. §14) of Alessandrini–Disarlo ([3, Conjecture 1.8]) in the case of unpunctured surfaces on the existence of stretch homeomorphisms with non-maximal stretch laminations.

Remark 1.9. In a subsequent paper [70], we use the existence and uniqueness of harmonic stretch lines to describe the "envelope" $\mathrm{Env}(X, Y)$ of Thurston geodesics from $X \in \mathcal{T}(S)$ to $Y \in \mathcal{T}(S)$.

1.4. Visual boundary of the Thurston metric and an exponential map

Finally, we extend the existence/uniqueness theory of harmonic stretch lines to rays whose terminal point is a projective measured lamination, representing a point on the boundary of the Thurston compactification of the Teichmüller space $\mathcal{T}(S)$. Properties of these rays, which are also Thurston geodesics, allow us to construct an "exponential" map on Teichmüller space from any base point $Y\in\mathcal{T}(S)$, with the boundary $\mathcal{PML}(S)$ of the Thurston compactification appearing as the visual boundary for this family of rays.

THEOREM 1.10. *For any $Y\in\mathcal{T}(S)$ and any $[\eta]\in\mathcal{PML}(S)$, there exists a unique harmonic stretch ray starting at Y, which converges to $[\eta]\in\mathcal{PML}(S)$ in the Thurston compactification.*

Moreover, these rays foliate $\mathcal{T}(S)$ if we fix Y and let $[\eta]$ vary in $\mathcal{PML}(S)$, or if we fix $[\eta]$ and let Y vary in $\mathcal{T}(S)$.

Remark 1.11. For any $X\in\mathcal{T}(S)$, the set of unit vectors tangent to Thurston stretch lines (whose stretch lamination is maximal) has Hausdorff measure zero in $T^1_X\mathcal{T}(S)$ ([90, Theorem 10.5]), while the set of unit vectors tangent to harmonic stretch lines is exactly $T^1_X\mathcal{T}(S)$ (see Proposition 13.12).

Remarks 1.12. So far, we have introduced several types of geodesics of the Thurston metric. (i) Harmonic stretch rays (lines) are special cases of piecewise harmonic stretch rays (lines). (ii) Those Thurston geodesics considered in Theorems 1.1, 1.3, 1.6 and 1.10 are harmonic stretch rays (lines) (see the second statement of Proposition 8.1 and Definition 12.1). (iii) For maximal geodesic laminations, piecewise harmonic stretch lines and Thurston stretch lines coincide (Lemma 7.10).

1.5. Geodesic flow for the Thurston metric

The Teichmüller geodesic flow on the moduli space has been extensively studied in the literature. However, the notion of geodesic flow does not exist naturally for the Thurston metric. A natural challenge by Rafi ([87, Problem 3.10]) is to introduce a suitable notion of geodesic flow for the Thurston metric. Here, we respond to this question by defining two versions of the geodesic flow for the Thurston metric.

Theorem 1.10 allows us to define the Thurston geodesic flow

$$\psi_t\colon\mathcal{T}(S)\times\mathcal{PML}(S)\longrightarrow\mathcal{T}(S)\times\mathcal{PML}(S)$$

such that the orbit through $(Y,[\eta])\in\mathcal{T}(S)\times\mathcal{PML}(S)$ is the harmonic stretch line determined by Y and $[\eta]$ via Theorem 1.10. Moreover, every harmonic stretch line appears as a (forward) orbit.

There is another version of the Thurston geodesic flow

$$\phi_t: \mathcal{T}(S) \times \mathcal{PML}(S) \longrightarrow \mathcal{T}(S) \times \mathcal{PML}(S)$$

such that the orbit through $(Y, [\eta]) \in \mathcal{T}(S) \times \mathcal{PML}(S)$ is the harmonic stretch line obtained from Theorem 1.1 as the limit of harmonic map rays $\mathbf{HR}_{X_t,Y}$ when X_t diverges along the harmonic map dual ray $\mathbf{hr}_{Y,\eta}$.

Both of these flows are mapping class group equivariant, and so they descend to $\mathcal{M}(S) \times \mathcal{PML}(S)$, where $\mathcal{M}(S)$ is the moduli space. Moreover, the earthquake flow and ϕ_t are compatible in the sense that earthquake translates of ϕ_t-orbits are still ϕ_t-orbits, and hence define an action of the upper-triangular subgroup of $\mathrm{SL}(2, \mathbb{R})$, see Proposition 13.14. We imagine it could be interesting to study the dynamical properties (invariant measures, ergodicity, and mixing) of these two versions of the Thurston geodesic flow over the moduli space.

1.6. Minimal graphs

An important tool for us will be conformal harmonic graphs over a hyperbolic domain with values in a real tree. Such maps will enjoy uniqueness properties which we can leverage to prove that the families of harmonic map rays, such as those in Theorem 1.1, have a unique limit for any chosen subsequence. The graphs we study have infinite diameter and are extensions to the singular setting of the properly embedded classical minimal surfaces studied by Jenkins–Serrin [44] (see also [19]). A complication is that because the trees do not, in general, fold, some of the usual arguments only partially generalize. Our principal result is the following theorem.

THEOREM 1.13. *Let Y be a crowned hyperbolic surface. Let T be an admissible dual tree of Y. Then, for any prescribed admissible partial boundary map there exists a unique $\pi_1(Y)$-equivariant minimal graph over $\widetilde{Y}$ in $\widetilde{Y} \times T$, where $\widetilde{Y}$ is the universal cover of Y.*

We define *admissible partial boundary map* just after Definition 5.4.

We apply the uniqueness portion of this result in our central results and so prove it within the body of the paper. Having opened the discussion of these sorts of graphs, we treat the existence portion in an appendix.

1.7. Extension of results from rays to disks

So far, we have focused on rays, i.e. sets defined in terms of a single real parameter. Yet an important topic in Teichmüller theory are Teichmüller disks, i.e. images of Teichmüller maps from X whose holomorphic quadratic differentials (on X) are proportional by a complex number.

On the hyperbolic geometric side, there are also distinguished disks, defined in terms of a measured geodesic lamination λ on a surface Y, and two operations that deform a hyperbolic structure that each use λ. Naturally, one may stretch the structure along the lamination, as we have described throughout this section. One can also perform an earthquake along this lamination. The two operations, defined with data X and a measured lamination λ, together define a real 2-dimensional family of hyperbolic surfaces through X, called a *stretch-earthquake disk*.

We state results that assert that the Hopf differential disks well-approximate stretch-earthquake disks for nearly degenerate domains X, and that there is a reasonable notion of dual Hopf differential disks that well-approximate Teichmüller disks for nearly degenerate ranges Y.

1.7.1. Convergence to Thurston stretch-earthquake disks

We begin with the approximation of stretch-earthquake disks.

Let $Y\in\mathcal{T}(S)$ be a hyperbolic surface and $\lambda\in\mathcal{ML}(S)$ a measured foliation/lamination. Let $\mathrm{PSL}_{Y,\lambda}$ be the piecewise harmonic stretch line obtained from Theorem 1.1 as the limit of harmonic map rays $\mathbf{HR}_{X_t,Y}$, where X_t diverges along the harmonic map dual ray $\mathbf{hr}_{Y,\lambda}$. Let $\mathcal{E}^s_\lambda(Y)$ be the surface obtained from Y by a time s earthquake along λ. Define the *stretch-earthquake disk* $\mathbf{SED}(Y,\lambda)$ of (Y,λ) to be the set

$$\bigcup_{-\infty<s<+\infty} \mathrm{PSL}_{\mathcal{E}^s_\lambda(Y),\lambda}(0,+\infty).$$

We wish to say, roughly, that the images of Hopf differential disks through a hyperbolic surface Y converge to a stretch-earthquake disk through Y — as we let the center X of the disks degenerate appropriately. To state this properly, we quickly introduce a bit more notation. Let $X_t\in\mathbf{hr}_{Y,\lambda}$ be the Riemann surface such that the horizontal foliation of $\mathrm{Hopf}(X_t,Y)=\Phi_t$ is $t\lambda$. Let $Y(t,r,s)$ be the hyperbolic surface such that

$$\mathrm{Hopf}(X_t,Y(t,r,s))=re^{is/2t}\Phi_t \quad\text{and}\quad Y_r=\mathrm{PSL}_{Y,\lambda}(r)\in\mathrm{PSL}_{Y,\lambda}\,.$$

For such data (Y,X_t,Φ_t), the *Hopf differential disk* $(\mathbf{HDD}(X_t,\Phi_t),Y)$ comprises surfaces Z satisfying $\mathrm{Hopf}(X_t,Z)=\zeta\Phi_t$ for some $\zeta\in\mathbb{C}$. (Naturally, $Y\in(\mathbf{HDD}(X_t,\Phi_t),Y)$ for the choice of $\zeta=1$ in the above.)

In this language, we may state our result on the convergence of disks in the harmonic setting to disks in the hyperbolic geometric setting.

Theorem 1.14. *Let $Y \in \mathcal{T}(S)$ be a hyperbolic surface and $\lambda \in \mathcal{ML}(S)$ be a measured foliation/lamination. Then, for X_t as above, the family of Hopf differential disks*

$$(\mathbf{HDD}(X_t, \Phi_t), Y)$$

with base point Y locally uniformly converge to the stretch-earthquake disk $(\mathbf{SED}(Y,\lambda), Y)$ with base point Y. Namely, for any prescribed $\mathbf{s}>0$ and $0<\mathbf{r}<\mathbf{r}'$, the family $Y(t,r,s)$ of surfaces converges to $\mathcal{E}^s_\lambda(Y_r)$ uniformly in $(r,s)\in[\mathbf{r},\mathbf{r}']\times[-\mathbf{s},\mathbf{s}]$ as $t\to\infty$.

1.7.2. Convergence to Teichmüller disks

On the other hand, we may look for a family of disks defined via harmonic maps that well-approximate Teichmüller disks. There are two possibilities for approximates, but we focus in this introduction on just the first: here we consider an S^1 family of harmonic map dual rays defined by a point Y very far from X in Teichmüller space, and then show that this "disk" of dual rays converges to a classical Teichmüller disk as Y diverges.

More precisely, begin with a surface Y and, fixing a domain X, a Hopf differential

$$\Phi = \mathrm{Hopf}(X, Y).$$

Let $Y_{s,\theta}$ be the hyperbolic surface such that

$$\mathrm{Hopf}(X, Y_{s,\theta}) = se^{2i\theta}\Phi.$$

In particular, $Y_{1,0}=Y$ and $Y_{s,\theta}$ diverges as $s\to\infty$ (for any choice of θ). Let

$$\mathbf{hd}_{X,\Phi,s} = \bigcup_{0\leqslant\theta\leqslant\pi} \mathbf{hr}_{Y_{s,\theta},\lambda_\theta}$$

denote the (variable target) harmonic map dual disk, where λ_θ is the horizontal foliation of $e^{2i\theta}\Phi$.

Let $\mathbf{TD}_{X,\Phi}$ be the Teichüller disk determined by X and Φ.

Theorem 1.15. *The harmonic map dual disk $\mathbf{hd}_{X,\Phi,s}$ locally uniformly converges to the Teichmüller disk $\mathbf{TD}_{X,\Phi}$, as $s\to+\infty$.*

In §10, we also provide for a version of convergence to Teichmüller horodisks.

1.8. Previous results

A number of authors have studied how harmonic maps might approximate Teichmüller maps, or how Thurston stretch maps might be approached from an analytic perspective.

Before Bers' exposition [5] of a proof of Teichmüller's theorem, Gerstenhaber and Rauch [29], [30] began a program to find Teichmüller maps as limits of energy-minimizing maps, optimized over conformal metrics. This program was completed by Mese in [60].

In the other direction, Daskalopoulos and Uhlenbeck [21], [20] focused on developing an analytic approach to finding least stretch maps: they build on literature on least gradient maps (see for example [84], [58], [85]), but they also develop the analytic foundations of what they term "J_p harmonic maps".

Bonsante, Mondello and Schlenker [9], [10] developed an S^1 action on $\mathcal{T}(S)\times\mathcal{T}(S)$, the "landslide flow", which limits to the earthquake flow when one of the parameters goes to a measured lamination in the Thurston boundary of $\mathcal{T}(S)$. Certainly, these landslides, as smooth approximates to earthquakes, have many interesting properties. Similarly to our analysis, these landslides $L(h,h^*)$ are defined in terms of the circle action on the Hopf differential disks, here centered at a Riemann surface c that is the "midpoint" of the ray between h and h^*; the limit to the earthquake flow on the second variable in $\mathcal{T}(S)\times\mathcal{T}(S)$ occurs as the first variable (not the midpoint) decays appropriately in Teichmüller space. In contrast, described in terms of the language of [9] and [10], here we study the limits of the Hopf differential disks as the center c degenerates along a dual ray; these are more specific paths than those studied by these authors, but allow us to show the convergence we require in our setting. The precise relationship between the two constructions is not clear.

There have been a number of studies comparing various rays in Teichmüller space. Choi, Rafi and Series show that short curves along lines of minima coincide with short curves along Teichmüller geodesics (with the same defining laminations) ([16]), and they prove that every line of minima is a Teichmüller quasi-geodesic ([17]); Choi, Dumas and Rafi prove that every grafting ray is a Teichmüller quasi-geodesic which stays within a bounded distance of some Teichmüller geodesic ([14]); Gupta proves that every grafting ray is asymptotic to a Teichmüller geodesic ([31], [32]); Lenzhen, Rafi and Tao compare the Teichmüller geodesics and Thurston geodesics in [52].

1.9. Organization of the paper

We provide a basic background and define our notation in §2 and §3. In §4 we find some subsequential limits of a family of harmonic maps $f_n\colon X_n\to Y$ from a degenerating sequence X_n of Riemann surfaces: crucial to our studies here and elsewhere is Minsky's [61] convex regions $\mathscr{P}_R$, and we include some first results here. In §5 we develop a key tool for our uniqueness and convergence claims: a type of Jenkins–Serrin uniqueness theorem for graphs over hyperbolic domains with values in a real tree. In §§6–8 we provide a proof of Theorem 1.1: subconvergence is proved in §6, we prove Theorem 1.5 in §7 and use that result and the uniqueness theorem for minimal graphs (from §5) to reach the final convergence result in §8.

We change foci in the next two sections. In §9 and §10 we consider limits of harmonic map dual rays when the range of the harmonic map degenerates, and find that Teichmüller rays and disks emerge: we prove Theorems 1.4 and 1.15 (as well as a second version of this convergence).

We then return to limits of the harmonic map rays and disks as the domain surface degenerates for the rest of the paper. In §11 we prove the convergence result (Theorem 11.3) on limits of families of harmonic map rays which define an earthquake-stretch disk. In §12, we prove our basic existence and uniqueness result for harmonic stretch lines, our refinement of Thurston geodesics for non-maximal laminations. In §13 and §14 we present consequences of our study, including a proof of Theorem 1.10 in §13.

The paper concludes by resolving a dangling issue: we prove in the appendix an existence result for our Jenkins–Serrin problem to complement the uniqueness result Theorem 5.7 that we used throughout.

1.10. Acknowledgements

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2. Definitions and background stretch maps and Teichmüller maps

2.1. Teichmüller space

Let S be an oriented closed surface of genus $g\geqslant 2$. A *marked Riemann surface* is a pair (X, f) where X is a Riemann surface and $f\colon S\to X$ is an orientation-preserving homeomorphism. Two marked Riemann surfaces (X, f) and (X', f') are called *equivalent* if there exists a conformal map in the homotopy class of $f'\circ f^{-1}$. The Teichmüller space $\mathcal{T}(S)$ is then defined as the space of equivalence classes of marked Riemann surfaces. Topologically, $\mathcal{T}(S)$ is homeomorphic to $\mathbb{R}^{6g-6}$.

The Teichmüller space can also be defined via hyperbolic surfaces. By the uniformization theorem, every Riemann surface of genus at least 2 admits a unique conformal metric of constant Gaussian curvature -1, the hyperbolic metric. A *marked hyperbolic surface* is a pair (X, f), where X is a hyperbolic surface and $f: S \to X$ is an orientation-preserving homeomorphism. Two marked hyperbolic surfaces (X, f) and (X', f') are called *equivalent* if there exists an isometry in the homotopy class of $f' \circ f^{-1}$. The Teichmüller space $\mathcal{T}(S)$ is then defined as the space of equivalence classes of marked hyperbolic surfaces.

For simplicity, we denote the (equivalence class of the) marked Riemann/hyperbolic surface (X, f) by X. Whenever we say a map from $X = (X, f)$ to $X' = (X', f')$, we mean it is a map homotopic to $f' \circ f^{-1}$, the change of marking. (An alternative perspective on equivalence is acquired by pulling X back to S by the homeomorphism f: then any equivalent pair (X, f) and (X', f') may be compared via the map $f^{-1} \circ f'$, which is homotopic to the identity map on S. We will often compare structures on S via maps of S that are homotopic to the identity.)

2.2. Quadratic differentials, measured foliations and measured laminations

Given a closed Riemann surface X of genus g, a holomorphic quadratic differential Φ is a holomorphic section of K_X^2, where K_X is the holomorphic cotangent bundle over X and K_X^2 is a tensor product of K_X. The space of holomorphic quadratic differentials on X, denoted by $H^0(X, K_X^2)$, is a vector space of real dimension $6g-6$.

Every holomorphic quadratic differential Φ defines two measured foliations, the *horizontal foliation* $\mathrm{Hor}(\Phi)$ by the imaginary part of $\sqrt{\Phi}$, and the *vertical foliation* $\mathrm{Vert}(\Phi)$ by the real part of $\sqrt{\Phi}$. Conversely, every measured foliation on X is realized as the horizontal/vertical foliation of a unique holomorphic quadratic differential on X (see [42] and [99]). The space of measured foliations on S, denoted by $\mathcal{MF}(S)$, is homeomorphic to $\mathbb{R}^{6g-6}$ (see for instance [26, Corollary 7.8]).

The hyperbolic counterpart of foliations are geodesic laminations. A *geodesic lamination* on a hyperbolic surface X is a closed subset which is a disjoint union of complete simple geodesics, called *leaves*. Typical examples are simple closed geodesics. A geodesic lamination λ is said to be *maximal* if the complementary regions on X are ideal triangles. Consider the space of geodesic laminations on S equipped with the Hausdorff topology of the space of closed subsets on X. This is a compact metric space with the Hausdorff distance. Since there is a canonical correspondence between the geodesics of any two marked hyperbolic metrics on a closed surface S, the space of geodesic laminations, denoted by $\mathcal{GL}(S)$, depends only on the topology of S. A collection of disjoint simple closed geodesics is called a *multicurve*. A geodesic lamination is called *chain-recurrent* if it can be approximated by multicurves in $\mathcal{GL}(S)$ in the Hausdorff metric on $\mathcal{GL}(S)$.

A *measured geodesic lamination* (or *measured lamination* for short) is a geodesic lamination equipped with a *transverse invariant measure*, which associates to every arc transverse to the lamination a Radon measure (see [8, pp. 11–13] for more details). Typical examples are simple closed geodesics with the Dirac measures. The *intersection number* between simple closed geodesics extends naturally to the setting of measured geodesic laminations as follows. Given a measured geodesic lamination λ and a simple closed geodesic α, the intersection number, denoted by $i(\lambda, \alpha)$, is defined to be the mass of α given by the transverse measure of λ. This associates to every measured geodesic lamination a function over the set of simple closed geodesics. Let $\mathcal{ML}(S)$ be the space of measured geodesic laminations, equipped with the weak-$*$ topology of the space of functions over the set of simple closed geodesics. Topologically, $\mathcal{ML}(S)$ is homeomorphic to $\mathbb{R}^{6g-6}$ (see [26, Theorem 6.1] or [74, Theorem 3.1.1]). There is a natural correspondence between the space of measure foliations and the space of measured geodesic laminations by straightening leaves of foliations using hyperbolic geodesics ([53]).

2.3. Extremal length and hyperbolic length

Given the Riemann surface X and a simple closed curve α, the *extremal length* $\mathrm{Ext}_X(\alpha)$ is defined as

$$\mathrm{Ext}_X(\alpha) = \sup_\rho \frac{\ell_\rho^2(\alpha)}{\mathrm{Area}(\rho)},$$

where the supremum ranges over all conformal metrics ρ on X, and where $\ell_\rho(\alpha)$ is the infimum of the length of curves (free) homotopic to α. Alternatively, we can also define the extremal length via the conformal modulus of cylinders. Any embedded cylinder C in X inherits a conformal structure from X and is conformal to a unique flat cylinder, up to a scaling. The *conformal modulus* (or *modulus*, for short) $\mathrm{Mod}(C)$ of C is defined as the height of flat cylinder divided by the circumference. Accordingly, we have

$$\mathrm{Ext}_C(\alpha) = \frac{1}{\mathrm{Mod}(C)},$$

where α is the core curve of C. For a Riemann surface X and any essential simple closed curve α on X, we have

$$\mathrm{Ext}_X(\alpha) = \inf_C \frac{1}{\mathrm{Mod}(C)}, \tag{2.1}$$

where the infimum ranges over all embedded cylinders with core curves homotopic to α (see [47, §3] for an explanation of (2.1)). In practice, one obtains a lower bound on the extremal length via the first formulation and an upper bound via the second formulation.

From the perspective of hyperbolic geometry, one can define the *hyperbolic length function* $\ell_X(\alpha)$ by associating to α the length of the unique geodesic representative with respect to the hyperbolic metric of X. Both the extremal length function and the hyperbolic length function can be extended to measured laminations and measured foliations ([47], [48]).

2.4. Teichmüller maps and Teichmüller rays

The Teichmüller map between a pair of Riemann surfaces is the solution to the variational problem of finding the minimizing quasiconformal constant within a given homotopy class. Let $X\in\mathcal{T}(S)$ be a Riemann surface and Φ a holomorphic quadratic differential on X. Near the regular points of Φ, there are natural coordinates in which one represents Φ as $dz^2=(dx+i\,dy)^2$. For any $k\geqslant 1$, consider the quadratic differential Φ_k locally defined by $(k^{1/2}\,dx+ik^{-1/2}\,dy)^2$. It defines a unique Riemann surface $X_k\in\mathcal{T}(S)$ on which Φ_k is holomorphic. The ray $\mathbf{TR}_{X,\Phi}\colon[1,\infty)\to\mathcal{T}(S)$ sending $k\geqslant 1$ to X_k is called a *Teichmüller ray*. (Note that the parametrization of the Teichmüller ray (as well as the Thurston stretch ray later in §2.6) is not the commonly used geodesic parametrization. The reason we use the current parametrization is to make parametrizations of various rays consistent with each other.) A *Teichmüller line* arises from allowing k in the construction of X_k to take on all positive values.

With respect to the natural coordinates of Φ and Φ_t, the identity map $X\to X_t$ is a Teichmüller map. Conversely, all Teichmüller maps arise in this way. Namely, for any two distinct Riemann surfaces $X,Y\in\mathcal{T}(S)$, there exist a unique holomorphic quadratic differential Φ on X (the *initial quadratic differential*) and a unique holomorphic quadratic differential Ψ on Y (the *terminal quadratic differential*), such that the (unique) Teichmüller map $X\to Y$ is locally defined by $(x,y)\mapsto(k^{1/2}x,k^{-1/2}y)$, with respect to the natural coordinates of Φ and Ψ, where $k>1$ is a positive constant (see [5], [27, Chapter 6] or [43, Chapter 5] for the existence and uniqueness of Teichmüller maps between Riemann surfaces of finite type). Consider the Teichmüller ray determined by X and the initial quadratic differential Φ. We denote it by $\mathbf{TR}_{X,Y}$, indicating that its initial point is at X and it passes through Y.

Given $X,Y\in\mathcal{T}(S)$, the Teichmüller distance d_T on $\mathcal{T}(S)$ is defined as

$$d_T(X,Y)=\frac{1}{2}\log K,$$

where K is the infimum of quasiconformal constants among quasiconformal maps from X to Y in the homotopy class determined by the markings of X and Y. Another formulation is given by Kerckhoff [47] in terms of extremal lengths:

$$d_T(X,Y) = \frac{1}{2}\log\sup_{\alpha}\frac{\mathrm{Ext}_Y(\alpha)}{\mathrm{Ext}_X(\alpha)},$$

where the supremum ranges over all simple closed curves. In this regard, the projective class of the horizontal foliation of the initial quadratic differential of the Teichmüller map from X to Y is characterized as the unique one realizing the maximum

$$\max_{\mu\in\mathcal{MF}(S)}\frac{\mathrm{Ext}_Y(\mu)}{\mathrm{Ext}_X(\mu)}.$$

Teichmüller lines are geodesics under the Teichmüller metric. Conversely, every geodesic under the Teichmüller metric is a Teichmüller line. Given a Teichmüller line $\mathbf{TR}_{X,\Phi}$, for any two points X_t and X_s in this line, we have

$$d_T(X_t,X_s) = \left|\log\sqrt{\frac{s}{t}}\right|.$$

2.5. Remark on ray and disk structures on Teichmüller space.

The paper concerns "ray structures" on Teichmüller space, where we imagine a number of different ways of defining the "rays". Here, by ray structure, we will mean that for every point $Y\in\mathcal{T}(S)$, there is a family of parameterized half-lines with initial point at Y which together foliate Teichmüller space. Probably the most famous such ray structure is that of Teichmüller rays from Y parameterized by rays in the vector space $H^0(Y,K_Y^2)$. That complex vector space $H^0(Y,K_Y^2)$ also has complex lines in it, and we imagine a "disk structure" on Teichmüller space $\mathcal{T}(S)$ to be an organization of a ray structure with some naturally defined free S^1 action on the rays.

2.6. Stretch maps

A seminal paper [90] defined a new (Finsler) metric ("Thurston's asymmetric metric") on Teichmüller space, characterized it in terms of the variational problem of finding the minimizing Lipschitz stretch between two hyperbolic surfaces, described a solution to that variational problem, and used the solution in describing shortest paths in Teichmüller space for the metric. In addition to Thurston's original paper, other sources for exposition of these topics, and further refinements of the theory, are [72], [15], [52], [93], [68], [22], [40], [71], [69].

A stretch map relies for its definition upon a maximal geodesic lamination λ on a hyperbolic surface. Recall that a maximal geodesic lamination has complementary regions which are ideal triangles. An ideal triangle admits a partial foliation by horocyclic arcs centered at the ideal points which both respects the symmetries of the ideal triangle and also meets the complete boundary of the ideal triangle. One defines a K-stretch map of the ideal triangle as a K-Lipschitz map which preserves the horocyclic foliation, fixes the three boundary points common to the foliations about each ideal point and stretches distance by K along the bounding geodesics (see [90, Proposition 3.2]).

Remarkably, Thurston's careful analysis shows that these stretch maps of complementary domains of the maximal geodesic lamination extend to a K-Lipschitz map between hyperbolic surfaces, and moreover, this extended map has the least possible Lipschitz constant among all maps in its homotopy class ([90, Corollary 4.2]).

We introduce some notation to summarize these constructions. If Y is a hyperbolic surface and λ is a maximal geodesic lamination, then we set $\mathbf{SR}_{Y,\lambda}(t)$ to define the surface constructed from Y and λ by a stretch map of Lipschitz constant $\sqrt{t}\geqslant 1$ along the lamination. We set $\mathbf{SR}_{Y,\lambda}(1,\infty)$ (resp. $\mathbf{SL}_{Y,\lambda}(0,\infty)$) to be the family of hyperbolic surfaces obtained by allowing t in $\mathbf{SR}_{Y,\lambda}(t)$ to range over all values in $[1,\infty)$ (resp. $(0,\infty)$), and we refer to that set as a *Thurston stretch ray* (resp. *Thurston stretch line*).

Given $X,Y\in\mathcal{T}(S)$, the Thurston (asymmetric) distance is defined by

$$d_{\mathrm{Th}}(X,Y) := \log L,$$

where L is the infimum of Lipschitz constants of Lipschitz homeomorphisms from X to Y in the homotopy class determined by the markings of X and Y. In particular, for $1\leqslant s\leqslant t$, we see that

$$d_{\mathrm{Th}}(\mathbf{SR}_{Y,\lambda}(s),\mathbf{SR}_{Y,\lambda}(t)) = \log\sqrt{\frac{t}{s}}.$$

Thurston also characterized this distance in terms of ratios of length functions of simple closed curves:

$$d_{\mathrm{Th}}(X,Y) = \log\sup_{\alpha}\frac{\ell_Y(\alpha)}{\ell_X(\alpha)}, \tag{2.2}$$

where the supremum ranges over all simple closed curves on X.

2.6.1. Maximally stretched laminations

As we mentioned earlier, for any two different Riemann surfaces $X,Y\in\mathcal{T}(S)$, the maximal ratio of extremal lengths

$$\max_{\mu\in\mathcal{MF}(S)}\frac{\mathrm{Ext}_Y(\mu)}{\mathrm{Ext}_X(\mu)}$$

is uniquely realized by the projective class of the horizontal measured foliation of the initial quadratic differential of the Teichmüller map from X to Y. For the ratio of hyperbolic length functions, the uniqueness is no longer true in the setting of measured laminations. If a non-uniquely ergodic measured lamination μ realizes the maximal ratio

$$\max_{\mu\in\mathcal{ML}(S)}\frac{\ell_Y(\mu)}{\ell_X(\mu)},$$

then any measured lamination with the same support as μ also realizes the maximal ratio. Nevertheless, Thurston showed that there is still a well-defined object in this setting, the *maximal maximally stretched chain-recurrent lamination* (or the *maximally stretched lamination* for short). For every two different hyperbolic surfaces X and Y, the maximally stretched lamination, denoted by $\Lambda(X,Y)$, is defined as the union of chain-recurrent laminations to which the restriction of every optimal Lipschitz map from X to Y is an affine map with the optimal Lipschitz constant.

THEOREM 2.1. ([90, Theorem 8.4]) *Let g and h be any two distinct hyperbolic structures on S. If g_i and h_i are sequences of hyperbolic structures converging to g and h, respectively, then $\Lambda(g,h)$ contains any lamination in the limit set of $\Lambda(g_i,h_i)$ in the Hausdorff topology.*

In particular, if $\Lambda(g_i,h_i)$ contains a simple closed curve α_i which converges to a maximal lamination λ in the Hausdorff topology, then $\Lambda(g,h)=\lambda$ because the only chain recurrent lamination containing λ is λ itself.

2.6.2. Generalized stretch maps and their rays

Not all of the laminations that will arise in our discussions will be maximal; indeed, a focus of this paper is the non-maximal case. Corresponding to the case when a quadratic differential will have higher-order zeros, we will find ourselves considering versions of stretch maps where the lamination will have complementary domains on a surface which are hyperbolic ideal polygons, or more generally, the so-called crowned hyperbolic surfaces. We extend the definition of stretch maps to this case, called *piecewise harmonic stretch maps* (see §7); we will also define in §12 a special class of Thurston geodesics which we call *harmonic stretch rays*.

2.7. Crowned hyperbolic surfaces

Definition 2.2. (Crown end) A *crown* with $m\geqslant 1$ boundary cusps is an open and incomplete hyperbolic surface bounded by a closed geodesic boundary c, and a *crown end* comprising bi-infinite geodesics $\{\gamma_i\}_{1\leqslant i\leqslant m}$ arranged in a cyclic order, such that the right half-line of the geodesic γ_i is asymptotic to the left half-line of geodesic γ_{i+1}, where $\gamma_{m+1}=\gamma_1$.

Definition 2.3. A *crowned hyperbolic surface* is obtained by attaching crowns to a compact hyperbolic surface with geodesic boundaries by isometries along some of their closed boundaries and removing the remaining geodesic boundaries. This results in an open and incomplete hyperbolic metric of finite area on the surface. Topologically, the underlying surface is the interior of a compact surface.

Remark 2.4. The definition of crowned hyperbolic surfaces here is slightly different from the definition in [34]. The definition in [34] requires all ends to be crown ends.

A *truncation* of a crown $\mathcal{C}$ with m boundary cusps is obtained from $\mathcal{C}$ by removing a collection of disjoint horodisk neighborhoods $U_1, U_2, ..., U_m$, one at each ideal vertex of $\mathcal{C}$.

Definition 2.5. The *metric residue* of the hyperbolic crown $\mathcal{C}$ with m boundary cusps is defined to be zero when m is odd, and equal to the absolute value of the alternating sum of lengths of geodesic sides of a truncation when m is even.

Remark 2.6. The metric residue does not depend on the choice of truncation.

2.8. Horizontal foliations of meromorphic quadratic differentials

Let Φ be a meromorphic quadratic differential on a compact Riemann surface X.

Definition 2.7. Let X and Φ be as above. A *saddle connection* is a finite $|\Phi|$-geodesic with endpoints at zeros or simple poles of Φ and whose interior points are regular points of Φ. A *horizontal saddle connection* of Φ is a segment of a horizontal leaf with endpoints on the zeros or simple poles of Φ. The *critical graph* of the horizontal foliation of Φ is the closure of the union of horizontal saddle connections of Φ and critical half-infinite leaves entering poles of Φ of order at least 2.

In particular, the critical graph of the horizontal foliation of Φ does not contain any recurrent (half-infinite) critical leaf of Φ. Consider a component X_0 complementary to the critical graph of Φ. Then, one of the following five possibilities occur ([86, §11.4]):

(i) X_0 is a horizontal cylinder of finite height foliated by closed horizontal leaves;

(ii) X_0 is precompact inside the complement in X of poles of Φ of order at least 2, but is not a horizontal cylinder;

(iii) X_0 is a half-infinite cylinder foliated by closed horizontal leaves;

(iv) X_0 is an infinite strip of finite height foliated by bi-infinite horizontal leaves;

(v) X_0 is a half-plane foliated by bi-infinite horizontal leaves.

Items (i) and (iii) correspond to [86, §11.4, type (1)]; item (ii) corresponds to [86, §11.4, type (2c)]; items (iv) and (v) correspond to [86, §11.4, type (2b)]. Finite cylinders in the first item and the closures of leaves in the second item are called *compact components* of Hor(Φ). Half-planes, strips and half-infinite cylinders are called *non-compact components* of Hor(Φ). Notice that a strip may spiral to a non-horizontal half-infinite cylinder, or be parallel to some half-plane, or both.

3. Harmonic maps

In this section, we provide a brief overview of harmonic maps between surfaces.

3.1. Harmonic maps

Let $(M, \sigma(z)|dz|^2)$ and $(N, \rho(w)|dw|^2)$ be two Riemannian surfaces. A differentiable map $f\colon M \to N$ is said to be *harmonic* if it satisfies the *Euler–Lagrange equation*

$$f_{z\bar{z}} + (\log\rho)_w f_z f_{\bar{z}} = 0.$$

If M and N are compact, then f is harmonic if and only if it is a critical point of the energy functional:

$$E(f) := \int_M e_f(z)\sigma(z)\, dz\, d\bar{z},$$

where

$$e_f(z) := \frac{\rho(f(z))}{\sigma(z)}(|f_z|^2 + |f_{\bar{z}}|^2).$$

Notice that the energy depends on the conformal structure on M and the metric on N. The energy of the map $f\colon X \to Y$ is denoted by

$$E(f) = E(f\colon X \to Y) = E(X, Y)$$

depending on the context that is required.

The basic existence result of harmonic maps in a homotopy class was established by Eells and Sampson in [24] and by Hamilton in [36], if the target manifold has non-positive sectional curvature. The uniqueness of harmonic maps in a homotopy class was obtained by Al′ber [2] and Hartman [37] if the target manifold has negative sectional curvature and if the image is not contractible to a point or a geodesic. Moreover, Sampson [75] and Schoen–Yau [78] proved that any harmonic map between compact surfaces which is homotopic to a diffeomorphism is a diffeomorphism, provided that the target surface has non-positive curvature.

3.2. Hopf differentials

Let X and Y be two hyperbolic surfaces. Let

$$f\colon (X,\sigma(z)\,|dz|^2)\longrightarrow (Y,\rho(w)\,|dw|^2) \tag{3.1}$$

be a harmonic diffeomorphism. Consider the pullback of ρ by f:

$$f^*(\rho)(z)=\rho(z)f_z\bar{f_{\bar z}}\,dz^2+e_f(z)\sigma(z)\,dz\,d\bar z+\rho(z)\bar{f_z}f_{\bar z}\,d\bar z^2.$$

The $(2,0)$-part of $f^*(\rho)$ is called the *Hopf differential* of f. The harmonicity of f implies that the Hopf differential of f is holomorphic (see [80], [45]). The Hopf differential for the map $f\colon X\to Y$ is denoted by

$$\mathrm{Hopf}(f)=\mathrm{Hopf}(f\colon X\to Y)=\mathrm{Hopf}(X,Y)=\Phi_X(Y)=\Phi,$$

depending on the context that is required.

Recall that every holomorphic quadratic differential defines two measured foliations on X, the horizontal foliation and the vertical foliation. The leaves of the horizontal foliation (resp. vertical foliation) of $\Phi:=\mathrm{Hopf}(f)$ are exactly the maximally stretched (resp. minimally stretched) directions of f. Choosing a local coordinate $z=x+iy$ on M such that the leaves of the horizontal foliation (resp. vertical foliation) of $\mathrm{Hopf}(f)$ are tangent to the x-axis (resp. y-axis). Then, in this coordinate, $\Phi=|\Phi|\,dz^2$. Hence,

$$f^*\rho=|\Phi|\,dz^2+e_f\sigma\,dz\,d\bar z+|\Phi|\,d\bar z^2=(e_f\sigma+2|\Phi|)\,dx^2+(e_f\sigma-2|\Phi|)\,dy^2. \tag{3.2}$$

In particular, if we choose the coordinate $z=x+iy$ such that $\Phi=dz^2$ and choose σ to be the singular flat metric induced by $|\Phi|$, then $f^*(\rho)$ can be simply expressed as

$$f^*\rho=(e_f+2)\,dx^2+(e_f-2)\,dy^2.$$

Let

$$\nu(z):=\frac{f_{\bar z}\,d\bar z}{f_z\,dz}$$

be the Beltrami differential of f. Set

$$\mathscr{G}(z)=\log\left(\frac{1}{|\nu(z)|}\right).$$

By calculation, we see that $\cosh\mathscr{G}=\sigma e_f/2|\Phi|$. Substituting this into (3.2) yields

$$(f^*\rho)(z)=2|\Phi(z)|(\cosh\mathscr{G}(z)+1)\,dx^2+2|\Phi(z)|(\cosh\mathscr{G}(z)-1)\,dy^2. \tag{3.3}$$

3.3. Harmonic maps to trees and minimal suspensions

We begin with a harmonic map $u: X \to Y$ with Hopf differential $\Phi = \mathrm{Hopf}(u)$. We lift the setting to the universal cover with a map $\tilde{u}: \widetilde{X} \to \widetilde{Y}$ and a Hopf differential $\tilde{\Phi} = \mathrm{Hopf}(\tilde{u})$. We consider the projection $p: \widetilde{X} \to T_h$ from $\widetilde{X}$ to the leaf space of the horizontal foliation $\mathrm{Hor}(\tilde{\Phi})$. That leaf space $T = T_h$, consisting of equivalence classes of points on the universal cover $\widetilde{S}$ of S, where two points are equivalent if they are contained in a connected leaf of $\mathrm{Hor}(\tilde{\Phi})$ (including leaves which branch at zeros of $\tilde{\Phi}$), has the structure of a tree, with topology induced from $\mathrm{Hor}(\tilde{\Phi})$. The tree acquires a well-defined distance $d = d_{T_h}$ from the push-forward $p_* \mu_{\tilde{\Phi}, h}$ of the lift $\mu_{\tilde{\Phi}, h}$ of the measure $\mu_{\Phi, h}$ on arcs transverse to the horizontal foliation of Φ. The metric tree (T, d) is not locally compact [99, §2.4] when the genus of X is at least 2, even if the vertices are typically of finite valence.

By construction, the map $p: \widetilde{X} \to (T_h, 2d)$ is harmonic in the senses of [99] (Definition 2.1) and [51, p. 643 and p. 656]. In the former case, it is a reflection that distances on a tree are submean near a vertex and proven (Proposition 3.1) within the paper, while in the latter case, it is an easy computation. In that case, the energy of the map to the tree is stationary, as a domain variation leads to a variation of energy expressed as the integrated product of the resulting infinitesimally trivial Beltrami differential for the domain variation against the Hopf differential of the projection, a pairing known to vanish by an integration by parts (cf. [1]); then, further, because the map p is a locally a projection, a local variation in the target along the map induces a domain variation, here using that because the tree is NPC, we may restrict to local variations that stay within the convex finite subtree that is the image of a small disk. Hence, as both

$$\tilde{u}: \widetilde{X} \longrightarrow \widetilde{Y} \quad \text{and} \quad p: \widetilde{X} \longrightarrow T_h$$

are harmonic, so is the product map

$$(\tilde{u}, p): \widetilde{X} \longrightarrow \widetilde{Y} \times T_h.$$

We call the latter product map the *minimal suspension* of the harmonic map u because, by construction, the product map $(\tilde{u}, p)$ is conformal, and since it is also harmonic, it is therefore minimal. The minimal suspension is stable in an appropriate sense (see [100] for this definition and further details), and indeed, given a tree (T, d) dual to a measured foliation $\mathcal{F}$, there is a unique minimal suspension

$$(\tilde{u}, p): \widetilde{X} \longrightarrow \widetilde{Y} \times (T, 2d)$$

so that both $\tilde{u}$ and p are harmonic with Hopf differentials that are additive inverses (so that $(\tilde{u}, p)$ is conformal and harmonic). A key portion of the present work, principally in §5 but also in the Appendix, may be seen as a less technically restrictive approach to the case where X is compact and also an extension to the case where X is complete but with punctures. Not formally related to any of these results but partially aligned in spirit is recent work of Markovic [55] in higher codimension.

To the equivariant harmonic map $p: \widetilde{X} \to (T_h, 2d)$ we associate the *equviariant energy* $E(X, T_h)$, defined as the integral of energy density over a fundamental domain on $\widetilde{X}$ of the fundamental group $\pi_1(X)$. Choose local coordinate $z=x+iy$ on $\widetilde{X}$ such that $\tilde{\Phi}=dz^2$. With respect to this coordinate, the map p is represented as $(x,y)\mapsto 2y$. Accordingly, the equivariant energy $E(X, T_h)$ is

$$E(X,T_h) = \int_{\widetilde{X}/\pi_1(S)} 2\,dx\,dy = 2\|\Phi\| = 2\,\mathrm{Ext}_X(\mathrm{Hor}(\Phi)), \tag{3.4}$$

where $\mathrm{Hor}(\Phi)$ is the horizontal measured foliation of Φ.

3.4. Harmonic map rays and dual rays

This paper centers on harmonic map rays in Teichmüller space as an interpolating structure between Thurston stretch rays and Teichmüller rays. In this subsection, we define these rays in two ways: the first is in terms of the Hopf differential, and the second is in terms of the minimal suspension in the previous subsection. The latter definition then suggests a dual construction of a different sort of ray structure which we call harmonic map dual rays.

3.4.1. Harmonic map rays

If $\Phi\in H^0(X, K_X^2)$ is a holomorphic quadratic differential on the Riemann surface X, then we are led to study the ray $s\Phi\subset H^0(X, K_X^2)$ for $s>0$. By the identification of the Teichmüller space $\mathcal{T}(S)$ with $H^0(X, K_X^2)$ via $Y\in\mathcal{T}(S)\mapsto \mathrm{Hopf}(X,Y)$ (cf. [96], [39]), we then obtain a family $Y_s\subset\mathcal{T}(S)$ of surfaces for which $\mathrm{Hopf}(X, Y_s)=s\Phi$ for some non-trivial element $\Phi\in H^0(X, K_X^2)$.

Definition 3.1. A family of hyperbolic surfaces Y_s for which $\mathrm{Hopf}(X, Y_s)=s\Phi$ for some non-trivial element $\Phi\in H^0(X, K_X^2)$ and $s\geqslant 0$ is called a *harmonic map ray* and is denoted by $\mathbf{HR}_{X,\Phi}(s)$, or sometimes by $\mathbf{HR}_{X,Y}(s)$, when we want to describe the ray passing between X and Y. In particular, $Y_0=X$.

We note that the ray $Y_s=\mathbf{HR}_{X,\Phi}(s)$ describes a family of minimal suspensions in $\widetilde{Y}_s\times(T_\Phi, 2s^{1/2}d)$ all with the same second factor T_Φ up to a scaling of the metric on the tree by $s^{1/2}$. In particular, we can imagine the ray as determining a change in the first factor $\widetilde{Y}_s$ of the two factors $\widetilde{Y}$ and T, so that the minimal surface's conformal structure $\widetilde{X}$ is held constant, while the second factor is scaled.

Regarding a minimal suspension $X\subset Y\times(T,2d)$ as having the three variables of the two factors Y and T, as well as the conformal structure X, we see that a new ray structure naturally presents itself: we may fix the first factor and the projective class of the second factor, and then let the conformal structure vary, as the scale of the tree varies.

Definition 3.2. The *harmonic map dual ray* $\mathbf{hr}_{Y,\Phi}(t)$ determined by a hyperbolic surface $Y\in\mathcal{T}(S)$ and holomorphic quadratic differential Φ is the family of conformal structures X_t such that

$$\mathrm{Hor}(\mathrm{Hopf}(X_t\to Y))=t\,\mathrm{Hor}(\Phi),$$

i.e. the horizontal foliation of the Hopf differential of the harmonic map from X_t to Y is proportional to that of Φ by a factor of $t\geqslant 0$. In particular, $X_0=Y$.

Indeed, the harmonic map dual ray has its parametrization defined only by the scaling of the dual tree to the horizontal measured foliation, say λ, of Φ. Thus, we often also use the notation $\mathbf{hr}_{Y,\lambda}(t)$ where λ is the horizontal measured foliation of Φ.

Equivalently, a harmonic map dual ray is a family of conformal structures defined via the minimal suspensions in a way dual to that of the harmonic map rays. The family $\mathbf{hr}_{Y,\Phi}(t)$ is the family of underlying conformal structures to the minimal graph in $Y\times(T_\Phi,2td)$ parameterized by t (we refer to [100, Theorem 3.1] for the existence and uniqueness of minimal graphs). Here, the duality is expressed as follows: for the harmonic map rays $\mathbf{HR}_{X,\Phi}(t)$, we fix the conformal structure of the minimal surface X and vary the surface $Y\in\mathbf{HR}_{X,\Phi}$, while for the dual ray $\mathbf{hr}_{Y,\Phi}(t)$, we fix the surface Y and let the conformal structure X vary in $\mathbf{hr}_{Y,\Phi}$.

To see that these rays provide a ray structure (cf. §2.5) for Teichmüller space $\mathcal{T}(S)$, see [100, Theorem 3.1] and [88, Theorem 1.3].

Our first focus in this paper will be the effect on the rays $\mathbf{HR}_{X,\Phi}(s)$ and $\mathbf{hr}_{Y,\Phi}(t)$ through a point on the ray if we allow a defining surface to degenerate along the other ray. More precisely, we fix a point $Y\in\mathcal{T}(S)$ and consider all the harmonic map rays $\mathbf{HR}_{X,\Phi}(s)$ which pass through Y and then study the limits of these as X tends to infinity along a harmonic map dual ray. Dually, we consider a fixed point X on a family of harmonic map dual rays and study the limits of those dual rays $\mathbf{hr}_{Y,\Phi}(t)$ as Y diverges along a harmonic map ray.

3.5. Analyticity of harmonic map rays

We recall the analyticity result of harmonic map rays in [98]. Before that, we need to introduce some notation. Let $X\in\mathcal{T}(S)$ and $\mathbf{HR}_{X,\Phi}\colon[0,\infty)\to\mathcal{T}(S)$ be the harmonic map ray defined by a holomorphic differential Φ on X. Consider the harmonic diffeomorphism $f_t\colon X\to\mathbf{HR}_{X,\Phi}(t)$ homotopic to the identity. In particular, the Hopf differential satisfies $\mathrm{Hopf}(f_t)=t\Phi$. Define the conformal energy density and anti-conformal energy density

$$\mathcal{H}_t:=\frac{\rho(f_t)}{\sigma}\left|\frac{\partial f_t}{\partial z}\right|^2 \quad\text{and}\quad \mathcal{L}_t:=\frac{\rho(f_t)}{\sigma}\left|\frac{\partial f_t}{\partial \bar z}\right|^2,$$

and the Laplacian

$$\Delta_\sigma:=\frac{4}{\sigma}\frac{\partial^2}{\partial z\,\partial\bar z}.$$

Then (see [96, p. 453]; see also [75] and [78])

$$e_t=\mathcal{H}_t+\mathcal{L}_t, \tag{3.5}$$

$$\frac{t^2|\Phi|^2}{\sigma^2}=\mathcal{H}_t\mathcal{L}_t, \tag{3.6}$$

$$\nu_t=\frac{\partial f_t/\partial\bar z}{\partial f_t/\partial z}=\frac{\overline{t\Phi}}{\sigma\mathcal{H}_t}, \tag{3.7}$$

$$|\nu_t|^2=\frac{\mathcal{L}_t}{\mathcal{H}_t}, \tag{3.8}$$

$$\Delta_\sigma\log\mathcal{H}_t=2\mathcal{H}_t-2\mathcal{L}_t+2K(\sigma)\qquad(\text{where }\mathcal{H}\neq 0), \tag{3.9}$$

$$\Delta_\sigma\log\mathcal{L}_t=2\mathcal{L}_t-2\mathcal{H}_t+2K(\sigma)\qquad(\text{where }\mathcal{L}\neq 0), \tag{3.10}$$

$$t\Phi=\sigma\mathcal{H}_t\bar\nu_t. \tag{3.11}$$

LEMMA 3.3. ([98, Theorem 2.2]) *With the notation as above, the conformal energy density $\mathcal{H}_t$ is real analytic in $t>0$.*

Proof. It is proved in [98, Theorem 2.2] that $\mathcal{H}_t$ is real analytic near $t=0$. With slight modification, that proof works for all $t\in(-\infty,+\infty)$. Indeed, for any $\alpha>0$, consider the functions space $C^{2,\alpha}(X)$ and the operator:

$$F\colon C^{2,\alpha}(X)\times\mathbb{C}\longrightarrow C^{2,\alpha}(X)$$

given by

$$F(\mathcal{H},t):=\Delta_\sigma\log\mathcal{H}-2\mathcal{H}+2t^2\frac{|\Phi|^2}{\sigma^2\mathcal{H}}+2.$$

For any $\tau\in(-\infty,+\infty)$, we have $F(\mathcal{H}_\tau,\tau)=0$, and the operator F is complex analytic near $(\mathcal{H}_\tau,\tau)$. Furthermore, by elementary computation, the linear operator $dF_{\mathcal{H}}$ at $(\mathcal{H}_\tau,\tau)$ is given by

$$\begin{aligned} dF_{\mathcal{H}}(\mathcal{H}_\tau,\tau)[\psi] &= \Delta_\sigma\frac{\psi}{\mathcal{H}_\tau}-2\psi-2\tau^2\frac{|\Phi|^2}{\sigma^2\mathcal{H}_\tau^2}\psi \\ &= \left\{\Delta_\sigma-2\mathcal{H}_\tau-2\tau^2\frac{|\Phi|^2}{\sigma^2\mathcal{H}_\tau}\right\}\left[\frac{\psi}{\mathcal{H}_\tau}\right] \\ &= \{\Delta_\sigma-2\mathcal{H}_\tau-2\mathcal{L}_\tau\}\left[\frac{\psi}{\mathcal{H}_\tau}\right] && \text{(by (3.6))} \\ &= \{\Delta_\sigma-2e_\tau\}\left[\frac{\psi}{\mathcal{H}_\tau}\right] && \text{(by (3.5))}. \end{aligned}$$

Since Δ_σ is a negative operator, $\mathcal{H}_t\geqslant 1$ ([96, Lemma 5.1]) and hence $e_\tau\geqslant 1$, we see, by the maximum principle as well as the bound $1\leqslant\mathcal{H}_\tau\leqslant C<\infty$ (for some bound C due to the compactness of X), that $dF_{\mathcal{H}}$ is invertible at $(\mathcal{H}_\tau,\tau)$. It then follows from the analytic implicit function theorem [4, p. 134, Theorem 3.3.2] that there exists a unique solution $\mathcal{H}=\mathcal{H}(t)$ of $F(\mathcal{H},t)$ near $(\mathcal{H}_\tau,\tau)$, and that $\mathcal{H}(t)$ is complex analytic near $t=\tau$. On the other hand, when t is real, the uniqueness of the solution implies that $\mathcal{H}(t)=\mathcal{H}_t$. In particular, we see that $\mathcal{H}_t$ is real analytic in t near $t=\tau$. The arbitrariness of τ then implies that $\mathcal{H}_t$ is real analytic in $t>0$. This completes the proof. □

As a direct consequence, we have the following result.

LEMMA 3.4. *Let $X\in\mathcal{T}(S)$ be a Riemann surface and Φ be a holomorphic quadratic differential on X. Then, the harmonic map ray*

$$\mathbf{HR}_{X,\Phi}\colon[0,\infty)\longrightarrow\mathcal{T}(S)$$

is real analytic in $t>0$.

Proof. Note that, by (3.1), the pullback on X of the hyperbolic metric of $\mathbf{HR}_{X,\Phi}(t)$ by f_t is

$$t\Phi(z)+e_t(z)\sigma(z)\,dz\,d\bar{z}+\overline{t\Phi(z)}, \tag{3.12}$$

where $e_t(z)=e_{f_t}(z)$ is the energy density of the harmonic map f_t and $\sigma(z)\,dz\,d\bar{z}$ is the hyperbolic metric of X. By (3.5) and (3.6),

$$e_t=\mathcal{H}_t+\frac{t^2|\Phi|^2}{\sigma^2\mathcal{H}_t}.$$

By Lemma 3.3, we see that $\mathcal{H}_t$ is real analytic in $t>0$. Therefore, the energy density e_t is also real analytic in $t>0$, so is the pullback metric on X of the hyperbolic metric on $\mathbf{HR}_{X,\Phi}(t)$. Accordingly, the harmonic map ray $\mathbf{HR}_{X,\Phi}(t)$ is real analytic in $t>0$. □

3.6. Minsky's estimates

The function $\mathscr{G}$ appearing in (3.3) is nearly zero at points which are far away from the zeros of Φ. More precisely, we have the following result.

LEMMA 3.5. ([61, Lemmas 3.2 and 3.3]) *Let X, Φ and $\mathscr{G}$ be as defined in §3.2. Let $p\in X$ be at a $|\Phi|$-distance at least d from any zero of Φ. Then,*

$$\mathscr{G}(p) \leqslant \frac{\sinh^{-1}(|\chi(M)|/d^2)}{\exp(d)}.$$

We also need the following estimate about the gradient of $\mathscr{G}$, with respect to the singular flat metric induced from $|\Phi|$.

LEMMA 3.6. *Let X, Φ and $\mathscr{G}$ be as defined in §3.2. Let $p\in X$ be at a $|\Phi|$-distance at least d from any zero of Φ. Then, there exists a constant c depending only on d and the topology of X such that*

$$|\nabla \mathscr{G}(p)| \leqslant c\mathscr{G}(p).$$

Proof. Recall that

$$\Delta \mathscr{G} = 4\sinh \mathscr{G} \tag{3.13}$$

(see for instance [61, equation (3.2)], where we choose $\sigma=|\Phi|$ and $K(\rho)=-1$). Combined with Lemma 3.5, this implies that

$$0 \leqslant \Delta \mathscr{G} \leqslant c_1 \mathscr{G}$$

for some constant depending only on d and the topology of X. By [38, Theorem 1], we see that there exists a constant c_2 depending only on d and c_1, and hence on d and the topology of X, such that

$$\mathscr{G}(q) \leqslant c_2 \mathscr{G}(p) \tag{3.14}$$

for any q in the $\frac{1}{2}d$ ball $B_{d/2}(p)$ centered at p under the $|\Phi|$-metric.

Next, we claim that there exists a constant c_3 depending only d such that

$$\sup_{q\in B_{d/4}(p)} |\nabla \mathscr{G}(q)| \leqslant c_3 \sup_{q\in B_{d/2}(p)} \mathscr{G}(q). \tag{3.15}$$

The proof of the claim follows almost line-by-line from the proof of [18, Theorem 2.1] for harmonic functions via the Bochner formula and testing functions. Equation (3.13) together with the fact that $\mathscr{G}>0$ ensure that

$$\langle \nabla \Delta \mathscr{G}, \nabla \mathscr{G}\rangle \geqslant 0 \quad \text{and} \quad \mathscr{G}\Delta \mathscr{G} \geqslant 0.$$

This is where the assumption of harmonicity is used in the proof of [18, Theorem 2.1]. The lemma then follows from (3.14) and (3.15). □

Minsky defined a family $\mathscr{P}_R$ of regions whose geometry is controlled and in whose complement the harmonic map is nearly a projection. We now summarize what we need of this work. Before stating the result, we need the notion of *boundary-convex*. Given a closed Riemann surface X and a holomorphic differential Φ on X, a subset C of X is *boundary-convex* if any geodesic arc, with respect to the singular flat metric induced from Φ, deformable rel endpoints into C, in fact lies in C. Equivalently, C is boundary-convex if the lift of each component of C to the universal cover $\widetilde{X}$ of X is convex under the singular flat metric induced from Φ. (We refer to [61, p. 172] for further explanation about boundary-convex.)

THEOREM 3.7. ([61, Theorem 5.1]) *Let X be a closed surface of genus g carrying a flat metric induced by a holomorphic quadratic differential Φ, and let $s>0$ and $c_1, ..., c_{3g-3}>0$ be chosen constants. Then, there exist positive constants A_1, K_1, K_2 and K_3, depending only on s and $\chi(X)$, and a nested family $\{\mathscr{P}_R\}_{R>0}$ of boundary-convex set $\mathscr{P}_R$ with the following properties*:

(i) *$\mathscr{P}_R$ contains the R neighbourhood of zeros of Φ.*

(ii) *Every component of $\partial\mathscr{P}_R$ is either polygonal comprising alternatively horizontal geodesic segments and vertical geodesic segments, or regular horizontal geodesics. The regular geodesic components occur in pairs bounding homotopically distinct flat cylinders.*

(iii) *If $\mathscr{F}_k$ is the k-th maximal horizontal cylinder whose subcylinders occur as components of $X\setminus\mathscr{P}_r$ for some $r\leqslant R$, and $\mathscr{F}_k$ is partially contained in $\mathscr{P}_R$ (that is, $\mathscr{F}_k$ itself is not contained in $\mathscr{P}_R$ but a subcylinder of $\mathscr{F}_k$ is contained in $\mathscr{P}_R$), then $\mathscr{F}_k\cap\mathscr{P}_R$ is a pair of flat cylinders with height at least $r_W+R+c_kR^2/W$, where $W=W(\mathscr{F}_k)$ is the circumference of $\mathscr{F}_k$ and*

$$r_W=W+\log\frac{1}{2}\sinh^{-1}\left(\frac{2|\chi(X)|}{W^2}\right).$$

(iv) *$\ell(\partial\mathscr{P}_R)\leqslant K_1R$.*

(v) *One has*

$$\operatorname{Area}(\mathscr{P}_R)\leqslant A_1+\left(K_2+2\sum_{i=1}^{k}c_i\right)R^2,$$

where k is the number of flat cylinder components of $M\setminus\mathscr{P}_r$ that have occurred for $r\leqslant R$.

(vi) *Each edge of a polygonal boundary component has length at least K_3R.*

(vii) *The polygonal components of $\partial\mathscr{P}_R$ are s-separated. Namely, for any two components γ_1 and γ_2 of $\partial\mathscr{P}_R$ (where possibly $\gamma_1=\gamma_2$), we have that any arc in X with endpoints in γ_1 and γ_2 that cannot be deformed (rel endpoints) into $\mathscr{P}_R$ has length greater than $s\max\{\ell(\gamma_1),\ell(\gamma_2)\}$.*

Before going further, let us make some comments about $\mathscr{P}_R$.

• Very roughly speaking, the basic strategy of the construction/proof of Theorem 3.7 is to start with the R-neighborhood of the zeros of Φ, boundary-convexify it, "square out the corners", and control the size of the resulting set. (Flat cylinders need separate treatment to satisfy property (iii).) In particular, we may assume that each component of $\mathscr{P}_R$ contains at least one zero of Φ.

• When R is small, say much less than half of the shortest $|\Phi|$-distance between zeros of Φ, the region $\mathscr{P}_R$ is a union of polygons, each surrounds a zero of Φ. As R increase, those polygons may merge, possibly resulting in new regular boundary geodesics. Eventually, when R is bigger than the $|\Phi|$-diameter of X, the region $\mathscr{P}_R$ is the whole surface X.

• From item (vii) of Theorem 3.7, for each polygonal boundary component $\partial_i\mathscr{P}_R$, one can append an embedded annulus of radius $\frac{1}{2}s\ell(\partial_i\mathscr{P}_R)$ outside $\mathscr{P}_R$. Furthermore, those appended annuli are pairwise disjoint. This is not necessary true for regular geodesic boundary components. On the other hand, both polygonal boundary components and regular geodesic boundary components admit annuli inside $\mathscr{P}_R$. Indeed, for regular geodesic boundary components, this is the content of item (iii) in Theorem 3.7; for polygonal boundary components, this is the content of Lemma 3.8 below.

LEMMA 3.8. *Let X, Φ, s, $c_1, \dots, c_{3g-3}$, K_3 and $\mathscr{P}_R$ be as in Theorem 3.7. Then, each polygonal boundary component $\partial_i\mathscr{P}_R$ of $\mathscr{P}_R$ admits an embedded annulus $\mathscr{A}_i$ in $\mathscr{P}_R$ such that the following conditions hold*:

(i) *$\partial_i\mathscr{P}_R$ is one of the two boundary components of $\mathscr{A}_i$*;

(ii) *the other component α' of $\partial\mathscr{A}_i$ is polygonal comprising alternatively horizontal geodesic segments and vertical geodesic segments parallel to the corresponding geodesic segments of $\partial_i\mathscr{P}_R$*;

(iii) *each point in α' is* (*exactly*) *at distance $\frac{1}{4}K_3'R$ from $\partial_i\mathscr{P}_R$, where*

$$K_3' = \min\{K_3, 1\};$$

(iv) *the angle at each corner of α' measured within $\mathscr{A}_i$ is $\frac{3}{2}\pi$.*

Proof. Let D be the infimum of the lengths of arcs in $\mathscr{P}_R$ that connect one point in $\partial_i\mathscr{P}_R$ to $\partial\mathscr{P}_R$, and are not homotopic rel endpoints to an arc in $\partial_i\mathscr{P}_R$. Since $\mathscr{P}_R$ is boundary convex, it follows that the minimum D is realized by (at least) one geodesic arc, say γ. Note that $\partial\mathscr{P}_R$ contains no zeros of Φ and comprises horizontal geodesic segments and vertical geodesic segments. Hence, the angle at each corner of $\partial\mathscr{P}_R$ is either $\frac{1}{2}\pi$ or $\frac{3}{2}\pi$. Being boundary convex implies that the angle at each corner of $\partial\mathscr{P}_R$, from the pointview of the interior of $\mathscr{P}_R$, is exactly $\frac{1}{2}\pi$. Hence, as a minimizing path in $\mathscr{P}_R$, the arc γ has to avoid the corners of $\partial\mathscr{P}_R$ and meet $\partial\mathscr{P}_R$ perpendicularly.

Next, we prove the following claim.

Claim 1. *There is a minimizing arc γ' realizing D that contains at least one zero of Φ.*

Proof. To see this, we start with the minimizing arc γ mentioned above. If γ itself contains a zero of Φ, then we are done. Suppose that γ does not contain any zero of Φ. Recall that γ meets $\partial\mathscr{P}_R$ perpendicularly. Therefore, the arc γ has to connect a horizontal segment of $\partial_i\mathscr{P}_R$ to a horizontal segment or connect a vertical segment to a vertical segment, as γ is a flat non-singular geodesic in a flat metric whose initial tangent is vertical (resp. horizontal), and thus is always vertical (resp. horizontal). From this, we see that there exists an embedded rectangle near γ which contains γ as one of its boundary sides, with γ connecting two other parallel boundary sides of $\partial\mathscr{P}_R$. We enlarge this rectangle by pushing outward the other boundary side $\widehat{\gamma}$ (that is parallel to γ) until it hits a corner of $\partial\mathscr{P}_R$ or a zero of Φ. If $\widehat{\gamma}$ hits a corner of $\partial\mathscr{P}_R$, then $\widehat{\gamma}$ contains a segment of $\partial\mathscr{P}_R$; thus, removing this subsegment yields a strictly shorter subarc that is homotopic to $\widehat{\gamma}$, and hence also homotopic to γ rel $\partial\mathscr{P}_R$. Since $\widehat{\gamma}$ and γ minimize distance, this contradicts the assumption that γ realizes the minimum D. Thus, the arc $\widehat{\gamma}$ must contain a zero of Φ. This establishes Claim 1 by taking $\widehat{\gamma}$ as the required γ'. □

From Claim 1 and item (i) of Theorem 3.7, we see that the length of γ' (or equivalently $\widehat{\gamma}$) is at least $2R$. Hence, the minimizing distance satisfies

$$D \geqslant 2R. \tag{3.16}$$

Let $I_1, I_2, ..., I_{2m}$ be the edges of the polygonal boundary component $\partial_i\mathscr{P}_R$, denoted in such a way that they appear consecutively. For each edge I_j of $\partial_i\mathscr{P}_R$, let $\square_j$ be the rectangle in $\mathscr{P}_R$ that contains I_j as one of its boundary sides and whose second pair of parallel sides (those on I_{j-1} and I_{j+1} in the usual notation) is also contained in $\partial_i\mathscr{P}_R$ and is of length $\frac{1}{2}K_3'R$ in total (i.e. of length $\frac{1}{4}K_3'R$ for each of the second pair of sides on I_{j-1} and I_{j+1}), where $K_3'=\min\{K_3, 1\}$ and K_3 is the constant from Theorem 3.7.

Claim 2. *We have that $\square_j\cap\square_k\neq\varnothing$ if and only if I_j and I_k are the two sides of a corner of $\partial_i\mathscr{P}_R$.*

Proof. To see this, for one direction, suppose that I_j and I_k are the two sides of a corner, then $\square_j\cap\square_k$ contains that corner, and hence is not an emptyset. For the converse direction, suppose that $\square_j\cap\square_k\neq\varnothing$, and let p be a point in $\square_j\cap\square_k$. Since three of the four sides of $\square_j$ are contained in $\partial_i\mathscr{P}_R$, it follows that $\square_j$ and $\square_k$ either coincide or share a common corner. The first possibility happens only if the component of $\mathscr{P}_R$ containing $\partial_i\mathscr{P}_R$ is itself a rectangle without any zero of Φ, which is excluded from our consideration (see the first comment following Theorem 3.7). Therefore, the two rectangles $\square_j$ and $\square_k$ must share a corner. Accordingly, the two edges I_j and I_k are the two sides of that corner in $\partial_i \mathscr{P}_R$. This proves Claim 2. $\square$

From Claim 2, we see that the union $\bigcup_j \square_j$ is a topological annulus with $\partial_i \mathscr{P}_R$ as one of its boundary components, and that the angle at each corner of the boundary component other than $\partial_i \mathscr{P}_R$ is exactly $\frac{3}{2}\pi$ measured within the annulus $\bigcup_j \square_j$. The lemma now follows by taking the union $\bigcup_j \square_j$ as the required annulus $\mathscr{A}_i$. $\square$

THEOREM 3.9. ([61, Theorem 7.1]) *Let $f: X \to Y$ be a harmonic diffeomorphism between closed hyperbolic surfaces of genus g with Hopf differential Φ. There are choices of constants $s>0$ and $c_1, \dots, c_{3g-3}>0$ for the construction of the polygonal region $\mathscr{P}_R$ and an $R_0>0$ such that in the complement of $\mathscr{P}_{R_0}$ there is a map π from the leaves of* $\mathrm{Hor}(\Phi)$ *to the lamination corresponding to $f_*(\mathrm{Hor}(\Phi))$ that factors through f, and is a local diffeomorphism on each leaf of* $\mathrm{Hor}(\Phi)$, *mapping it to the corresponding geodesic representative of its image. For any point p on a leaf in $X \setminus \mathscr{P}_{R_0}$,*

$$d_Y(f(p), \pi(p)) < a \exp(-b d_{|\Phi|}(p, \mathscr{P}_{R_0})),$$

where d_Y is the hyperbolic distance on Y, and the derivative of π along leaves with respect to the $|\Phi|$-metric satisfies

$$\big| |d\pi| - 2 \big| \leqslant a \exp(-b d_{|\Phi|}(p, \mathscr{P}_{R_0})),$$

where a and b are positive constants depending only on $\chi(X)$.

THEOREM 3.10. ([61, Theorem 7.2]) *There exists a constant C_0 depending on the topology of S such that, for every $X, Y \in \mathcal{T}(S)$,*

$$\sup_{\mu \in \mathcal{ML}(S)} \frac{1}{2} \frac{\ell_Y^2(\mu)}{\mathrm{Ext}_X(\mu)} \leqslant E(X,Y) \leqslant \frac{1}{2} \frac{\ell_Y^2(\gamma)}{\mathrm{Ext}_X(\gamma)} + C_0 \tag{3.17}$$

where γ is the horizontal measured foliation of the Hopf differential of the harmonic diffeomorphism $f: X \to Y$ homotopic to the change of marking.

Remark 3.11. The method of Minsky actually proves a slightly stronger estimate in the following sense. Let Φ be the Hopf differential of the harmonic diffeomorphism $f: X \to Y$ with γ being the horizontal measured foliation/lamination. Let $\{X_i\}$ be the components complementary to the critical graph (see Definition 2.7) of (the horizontal foliation γ of) Φ and let γ_i be the restriction of γ to X_i. Since each X_i is locally convex and contains every leaf of γ_i, the construction of Minsky's polygonal region in

Theorem 3.7 and train-track approximate in the proof of Theorem 3.9 occur within each X_i (cf. especially [61, §5 and §6]). Then, Theorem 3.10 holds on X_i. Namely,

$$\sup_{\mu\in\mathcal{ML}(X_i)} \frac{1}{2}\frac{\ell_Y^2(\mu)}{\operatorname{Ext}_{X_i}(\mu)} \leqslant E(f|_{X_i}) \leqslant \frac{1}{2}\frac{\ell_Y^2(\gamma_i)}{\operatorname{Ext}_{X_i}(\gamma_i)}+C_0, \tag{3.18}$$

where $\mathcal{ML}(X_i)$ represents the measured foliations/laminations on X that can be homotoped into X_i

Recall [61, p. 165, equation (3.4)] (see also [96, Lemma 3.2]) that, for any subsurface $U\subset X$, the energy and norm of Hopf differential Φ satisifes

$$2\|\Phi|_U\| \leqslant E(f|_U) \leqslant 2\|\Phi|_U\|+2\pi|\chi(S)|. \tag{3.19}$$

Notice that, for the horizontal measured foliation/lamination γ of Φ,

$$\operatorname{Ext}_X(\gamma)=\|\Phi\|. \tag{3.20}$$

Combining (3.17), (3.19), and (3.20), we obtain some basic first estimates for this paper.

Lemma 3.12. *For any closed orientable surface S of genus at least 2 there exists a constant C depending only on S such that the following holds. Let $X,Y\in\mathcal{T}(S)$ be two hyperbolic surfaces and $f\colon X\to Y$ the unique harmonic map homotopic to the change of marking. Let Φ be the Hopf differential of f. Let γ be the measured lamination corresponding to the horizontal measured foliation of Φ. Then,*

$$\ell_Y(\gamma)-C\leqslant 2\|\Phi\|\leqslant \ell_Y(\gamma)+C \tag{3.21}$$

and

$$\ell_Y(\gamma)-C\leqslant E(f)\leqslant \ell_Y(\gamma)+C. \tag{3.22}$$

Moreover, let $\gamma=\sum_{i=1}^k \gamma_i$ be the component decomposition of γ. Let X_i be the component supporting γ_i in the complementary region of the critical graph (*see Definition* 2.7) *of Φ. Then,*

$$\ell_Y(\gamma_i)-C\leqslant 2\|\Phi|_{X_i}\|\leqslant \ell_Y(\gamma_i)+C \tag{3.23}$$

and

$$\ell_Y(\gamma_i)-C\leqslant E(f|_{X_i})\leqslant \ell_Y(\gamma_i)+C. \tag{3.24}$$

Proof. It follows from (3.17), (3.19) and (3.20) that

$$4\|\Phi\|\left(\|\Phi\|-\frac{C_0}{2}\right)\leqslant \ell_Y^2(\gamma)\leqslant 4\|\Phi\|(\|\Phi\|+\pi|\chi(S)|),$$

which implies (3.21). Inequality (3.22) then follows from (3.19) and (3.21). Upon replacing (3.17) by (3.18) in the above reasoning, we obtain (3.23) and (3.24). □

4. Compactness of harmonic maps to a fixed target

The goal of this section is to prove a compactness result for harmonic maps to a fixed target (Lemma 4.6). We continue with the notation introduced in the previous section.

4.1. Extension of estimates on Minsky's polygonal regions

We begin with an extension of Minsky's analysis of the polygonal region that we will need in our discussion of the compactness of a family of Riemann surface domains for the harmonic maps.

LEMMA 4.1. *Let $Y\in\mathcal{T}(S)$ be a hyperbolic surface. For any R sufficiently large, there exist two positive constants δ and ρ depending on R and Y such that, for any harmonic map $f\colon X\to Y$ with $X\in\mathcal{T}(S)$ homotopic to the change of marking, each component of the polygonal region $\mathscr{P}_R$ has injectivity radius at least δ and diameter at most ρ with respect to the Hopf differential metric.*

Remark 4.2. This lemma will be important in controlling the shape of limits of harmonic maps. As one simple example, note that this lemma means that one cannot pinch the domain along short curves containing a zero of the Hopf differential: the zeros are always in the interior of the Minsky regions, and the Lemma states that these regions have injectivity radii bounded away from zero.

Proof of Lemma 4.1. If $\mathscr{P}_R$ is a topological disk, then the conclusion follows directly from Minsky's construction. Now let us assume that $\mathscr{P}_R$ is not a topological disk.

CLAIM 1. *There exists a positive constant δ_1 depending on R and Y such that the extremal length of any essential simple closed curve α of $\mathscr{P}_R$ is at least δ_1:*

$$\operatorname{Ext}_{\mathscr{P}_R}(\alpha)\geqslant\delta_1.$$

Proof. To see this first claim, note that by (3.19), we have

$$E(f|_{\mathscr{P}_R})\leqslant 2\|\operatorname{Hopf}(f)\|_{\mathscr{P}_R}+\operatorname{Area}(Y)\leqslant cR^2+\operatorname{Area}(Y).$$

Then, it follows from (3.17) that

$$\operatorname{Ext}_{\mathscr{P}_R}(\alpha)\geqslant\frac{\ell_Y^2(\alpha)}{2E(f|_{\mathscr{P}_R})}\geqslant\frac{\operatorname{Syst}(Y)^2}{2cR^2+2\operatorname{Area}(Y)}.$$

This proves Claim 1. □

Next, we claim the following.

Claim 2. *There exists a positive constant δ_2 depending on the topology of S such that, for each boundary component α of $\mathscr{P}_R$, we have* $\mathrm{Ext}_{\mathscr{P}_R}(\alpha)\leqslant\delta_2$.

Proof. To see this second claim, note that there are two cases here, depending on whether the given boundary component α is polygonal or regular. We start with the case where α is a regular geodesic. By items (iii) and (iv) of Theorem 3.7, α admits a horizontal cylinder $\mathscr{C}$ inside $\mathscr{P}_R$ of height at least R and of circumference $\ell(\alpha)$ at most K_1R, for some positive constant K_1 depending the topology of S. Hence, the modulus $\mathrm{Mod}(\mathscr{C})$ satisfies

$$\mathrm{Mod}(\mathscr{C})\geqslant\frac{R}{\ell(\alpha)}\geqslant\frac{1}{K_1}.$$

According to (2.1), we see that

$$\mathrm{Ext}_{\mathscr{P}_R}(\alpha)\leqslant\frac{1}{\mathrm{Mod}(\mathscr{C})}\leqslant K_1. \tag{4.1}$$

Now we turn to the second case where α is a polygonal boundary component. Let $\mathscr{A}\subset\mathscr{P}_R$ be the annulus attached to α obtained from Lemma 3.8. In particular, the annulus $\mathscr{A}$ contains α as one of its boundary components. Suppose that α has m corners. Then, it has m horizontal segments $\{h_1,...,h_m\}$ and m vertical segments $\{v_1,...,v_m\}$. By items (ii) and (iii) of Lemma 3.8, we infer that the other boundary component α' of the annulus $\mathscr{A}$ consists of m horizontal segments $\{h'_1,...,h'_m\}$ with lengths

$$\ell(h'_i)=\ell(h_i)-\frac{K'_3R}{2},$$

m vertical segments $\{v'_1,...,v'_m\}$ with lengths

$$\ell(v'_i)=\ell(v_i)-\frac{K'_3R}{2}.$$

Therefore, the lengths of α' and α satisfy

$$\ell(\alpha')=\sum_i(\ell(h'_i)+\ell(v'_i))=\sum_i\left(\ell(h_i)-\frac{K'_3R}{2}+\ell(v_i)-\frac{K'_3R}{2}\right)=\ell(\alpha)-mK'_3R.$$

From item (iii) of Lemma 3.8, it follows that the annulus $\mathscr{A}$ contains the regular annulus $\mathscr{A}'$ of α' of radius $\frac{1}{4}K'_3R$:

$$\mathscr{A}':=\left\{p\in\mathscr{A}:d(p,\alpha')=\tfrac{1}{4}K'_3R\right\},$$

where d refers to the singular $|\Phi|$-metric. By item (iv) of Lemma 3.8, the angle at each corner of α' with respect to $\mathscr{A}$ is $\frac{3}{2}\pi$. So the total curvature $\mathbb{K}_{\alpha'}$ of α' with respect to both

$\mathscr{A}$ and $\mathscr{A}'$ is $\mathbb{K}_{\alpha'} = -\frac{1}{2}m\pi$. The boundary component α'' of $\mathscr{A}'$ other than α' consists of m horizontal segments $\{h''_1, ..., h''_m\}$ with lengths $\ell(h''_i) = \ell(h'_i)$, and m vertical segments $\{v''_1, ..., v''_m\}$ with lengths $\ell(v''_i) = \ell(v'_i)$, and m circular arcs centered at each corner of α' of radius $\frac{1}{4}K'_3R$ and of angle $\frac{1}{2}\pi$. Hence, the length $\ell(\alpha'')$ satisfies

$$\ell(\alpha'') = \sum_i (\ell(h''_i) + \ell(v''_i)) + m \cdot \frac{\pi}{2} \cdot \frac{K'_3R}{4} = \ell(\alpha') + \frac{mK'_3R\pi}{8}. \tag{4.2}$$

Combined with [61, Theorem 4.5], this implies that the modulus of $\mathscr{A}'$ satisfies

$$\begin{aligned}
\mathrm{Mod}(\mathscr{A}') &\geqslant \frac{1}{|\mathbb{K}_{\alpha'}|} \log \frac{\ell(\alpha'')}{\ell(\alpha')} && \text{(by [61, Theorem 4.5])} \\
&= \frac{1}{m\pi/2} \log \frac{\ell(\alpha') + mK'_3R\pi/8}{\ell(\alpha')} \\
&\geqslant \frac{1}{m\pi/2} \log \frac{mK'_3R\pi}{8\ell(\alpha')} \\
&= \frac{1}{m\pi/2} \log \frac{mK'_3R\pi}{8(\ell(\alpha) - mK'_3R)} && \text{(by (4.2))} \\
&\geqslant \frac{1}{m\pi/2} \log \frac{mK'_3R\pi}{8\ell(\alpha)} \\
&\geqslant \frac{1}{m\pi/2} \log \frac{mK'_3\pi}{8K_1} && \text{(by Theorem 3.7 (iv))}.
\end{aligned}$$

Recall that the angle at each corner of each polygonal boundary component of $\mathscr{P}_R$ with respect to the interior of $\mathscr{P}_R$ is $\frac{1}{2}\pi$. Applying the Gauss-Bonnet formula to $\mathscr{P}_R$ with the singular $|\Phi|$-metric, we see that total number of corners of $\partial\mathscr{P}_R$, hence also the number m of corners of α, is bounded from above by some constant $\mathbf{k}$ depending only the topology of the underlying surface X, that is,

$$m = \#\{\text{corners of } \partial\mathscr{P}_R\} \leqslant \mathbf{k}. \tag{4.3}$$

Inserting this fact into the above-displayed equation about moduli, we see that

$$\mathrm{Mod}(\mathscr{A}') \geqslant C$$

for some positive constant C depending on the topology of X. Therefore, by (2.1), the extremal length of α on $\mathscr{P}_R$ satisfies:

$$\mathrm{Ext}_{\mathscr{P}_R}(\alpha) \leqslant \frac{1}{\mathrm{Mod}(\mathscr{A}')} \leqslant \frac{1}{C}$$

for some constant C depending only on the topology of X. Combined with (4.1), this establishes Claim 2. □

Thirdly, let us consider the double of $\mathscr{P}_R$, denoted by $\mathscr{P}_R^d$ obtained by gluing a copy of $\mathscr{P}_R$ to (the oppositely oriented) $\mathscr{P}_R$ along the corresponding boundary edges.

Claim 3. *There exists a positive constant δ_3 such that, for any essential simple closed curve α on $\mathscr{P}_R^d$, we have $\mathrm{Ext}_{\mathscr{P}_R^d}(\alpha)\geqslant\delta_3$.*

Proof. If α is a boundary component of $\mathscr{P}_R$ or an interior curve of $\mathscr{P}_R$, then

$$\mathrm{Ext}_{\mathscr{P}_R^d}(\alpha)\geqslant\frac{1}{2}\,\mathrm{Ext}_{\mathscr{P}_R}(\alpha)\geqslant\frac{\delta_1}{2}.$$

If α intersects one of the boundary components of $\mathscr{P}_R$ say β, then

$$\mathrm{Ext}_{\mathscr{P}_R^d}(\alpha)\geqslant\frac{i(\alpha,\beta)^2}{\mathrm{Ext}_{\mathscr{P}_R^d}(\beta)}\geqslant\frac{1}{\mathrm{Ext}_{\mathscr{P}_R^d}(\beta)}\geqslant\frac{1}{\mathrm{Ext}_{\mathscr{P}_R}(\beta)}\geqslant\frac{1}{\delta_2},$$

where the last inequality follows from the second claim above. This proves Claim 3. □

Finally, we claim that there exist $\delta_4>0$ and $\rho>0$ depending only on R and the topology of S such that the following claim holds.

Claim 4. *There exist constants δ_4 and ρ such that the polygonal region $\mathscr{P}_R$ has injectivity radius at least δ_4 and diameter at most ρ with respect to the Hopf differential metric.*

Proof. Notice that the restriction of the Hopf differential to $\mathscr{P}_R$ and its copy gives a meromorphic quadratic differential q^d on $\mathscr{P}_R^d$. This differential has simple poles at the corners of the boundary components of $\mathscr{P}_R$; the number of these points is bounded from above by some constant $\mathbf{k}$ depending only on the topology of S (see (4.3)). Notice that the genus of $\mathscr{P}_R^d$ is at most

$$\mathrm{genus}(X)+(3\,\mathrm{genus}(X)-2)=4\,\mathrm{genus}(X)-2.$$

Set $\mathbf{g}=4\,\mathrm{genus}(X)-2$. Then,

$$\left(\mathscr{P}_R^d,\frac{q^d}{2\|\mathscr{P}_R\|}\right)\in\bigcup_{\substack{g\leqslant\mathbf{g}\\ \kappa\leqslant\mathbf{k}}}\mathcal{Q}^\kappa\mathcal{M}_g,$$

where $\mathcal{Q}^\kappa\mathcal{M}_g^\kappa$ represents the bundle over the moduli space $\mathcal{M}_g$ whose fiber, over a Riemann surface $M\in\mathcal{M}_g$, is the space $Q^\kappa(M)$ of area one meromorphic quadratic differentials having κ simple poles. Moreover, combined with Mumford's compactness theorem [65, Corollary 3] and an inequality between hyperbolic length and extremal length [56, Corollary 3], the third claim above implies that for each g there exists a compact subset $K_g \subset \mathcal{M}_g$ such that $\mathscr{P}_R^d \subset K_g$, i.e.

$$\left(\mathscr{P}_R^d, \frac{q^d}{2\|\mathscr{P}_R\|}\right) \in \bigcup_{\substack{g \leqslant \mathbf{g} \\ \kappa \leqslant \mathbf{k}}} \mathcal{Q}^\kappa K_g.$$

Note that, in general, simple poles may collide to form less singular points, so for any κ, neither $\mathcal{Q}^\kappa(M)$ nor $\mathcal{Q}^\kappa K_g$ is compact. On the other hand, simple poles cannot collide to form poles of higher order because of the assumption of area one, so they could only collide to form regular points or zeros. Therefore, both

$$\bigcup_{0 \leqslant \kappa \leqslant \mathbf{k}} \mathcal{Q}^\kappa(M) \quad \text{and} \quad \bigcup_{0 \leqslant \kappa \leqslant \mathbf{k}} \mathcal{Q}^\kappa K_g$$

are compact (as a subcomplex of the stratified space of quadratic differentials of unit area and with at most simple poles). Then, for each $g \leqslant \mathbf{g}$, there exist two positive constants δ_g and ρ_g such that

$$\inf_{\substack{0 \leqslant \kappa \leqslant \mathbf{k} \\ (M,q') \in \mathcal{Q}^\kappa K_g(M)}} \frac{\operatorname{inj}(q')}{\operatorname{inj}(M)} \geqslant \delta_g \quad \text{and} \quad \sup_{\substack{0 \leqslant \kappa \leqslant \mathbf{k} \\ (M,q') \in \mathcal{Q}^\kappa K_g(M)}} \frac{\operatorname{diam}(q')}{\operatorname{diam}(M)} \leqslant \rho_g,$$

where $\operatorname{inj}(M)$ and $\operatorname{diam}(M)$ represent respectively the injectivity radius and diameter of the hyperbolic metric of M, and where $\operatorname{inj}(q')$ and $\operatorname{diam}(q')$ represent respectively the injectivity radius and diameter of the singular flat metric induced by q'. Hence, for any $M \in K_g$ and any $q' \in \bigcup_{0 \leqslant \kappa \leqslant \mathbf{k}} Q^\kappa(M)$, we have

$$\operatorname{inj}(q') \geqslant \delta_g \min_{M \in K_g} \operatorname{inj}(M) \qquad \text{and} \qquad \operatorname{diam}(q') \leqslant \rho_g \max_{M \in K_g} \operatorname{inj}(M).$$

Then, $0 \leqslant g \leqslant \mathbf{g} < \infty$ implies that

$$\delta_4' := \min_{0 < g \leqslant \mathbf{g}} \delta_g \min_{M \in K_g} \operatorname{inj}(M) > 0,$$

and

$$\rho_4' := \max_{0 < g \leqslant \mathbf{g}} \rho_g \max_{M \in K_g} \operatorname{inj}(M) < \infty.$$

In particular, the injectivity radius of $q^d/2\|\mathscr{P}_R\|$ with respect to the singular flat metric $q^d/2\|\mathscr{P}_R\|$ is at least δ_4'. Hence, the injectivity radius of q^d is at least $2\|\mathscr{P}_R\|\delta_4'$, which is at least $3\pi R^2 \delta_4'$, because $\mathscr{P}_R$ contains the R neighbourhood of its zeros. Recall that q^d is obtained as a double of $\mathscr{P}_R$ along its boundary edges. Therefore, the injectivity radius of $\mathscr{P}_R$ is at least $\frac{3}{2}\pi R^2 \delta_4'$. Similarly, the diameter of $\mathscr{P}_R$ is at most $2\|\mathscr{P}_R\|\rho_4'$. Together with item (v) of Theorem 3.7, this implies that the diameter of $\mathscr{P}_R$ is bounded from above by some constant depending on R and the topology of S. □

This finishes the proof of Lemma 4.1. □

We will need an estimate on the total error that accumulates in using that the harmonic map outside $\mathscr{P}_R$ is a projection.

LEMMA 4.3. *Let R_0 be the constant from Theorem 3.9. Let $f\colon X\to Y$ be a harmonic diffeomorphism between closed hyperbolic surfaces $X,Y\in\mathcal{T}(S)$. Let $e(f)$ be the energy density with respect to the metric induced by the Hopf differential* Hopf(f). *Then, for any $R>R_0$,*

$$\int_{X\setminus\mathscr{P}_R} |e(f)-2|\,dA \leqslant C|\chi(S)|e^{-R/2},$$

where C is a constant depending on $\chi(X)$, $\mathscr{P}_R$ is Minsky's polygonal region and dA is the area measure induced by the Hopf differential Hopf(f).

Proof. Let R_0 be the constant from Minsky's estimate. Then, by equation (3.3) and Lemma 3.5, for any $R>R_0$ and any $p\in X\setminus\mathscr{P}_R$, we have

$$|e(f)-2| \leqslant Ce^{-\operatorname{dist}(p,\mathscr{Z})}, \tag{4.4}$$

where C is a constant depending on $\chi(X)$, and $\operatorname{dist}(p,\mathscr{Z})$ represents the distance from p to the zero set $\mathscr{Z}$ of Hopf(f) with respect to the flat metric induced by Hopf(f).

Consider the Voronoi decomposition of X with respect to Hopf(f), where the 2-cells are the path components of the set of points which have unique length-minimizing paths to the zero set of Hopf(f). The number of such 2-cells is exactly the number of zeros of Hopf(f) counted without multiplicity. Within each 2-cell, consider the horizontal critical segments initiating from the corresponding zero of Hopf(f). Let κ be the order of this zero. These segments cut the underlying 2-cell into $\kappa+2$ sub-cells, each of which can be identified with a subset of the upper or lower half-plane in $\mathbb{C}$. The total number of such sub-cells is at most $3(4g-4)$, which corresponds to the case where all zeros of Hopf(f) are simple. Integrating $|e(f)-2)|$ over each sub-cell, we obtain

$$\begin{aligned}\int_{X\setminus\mathscr{P}_R} |e(f)-2|\,dA &\leqslant 3(4g-4)C\int_{\{z\in\mathbb{C}:|z|\geqslant R,\operatorname{Im} z\geqslant 0\}} e^{-|z|}\,dA\\ &\leqslant 6(2g-2)C\pi e^{-R/2}.\end{aligned}$$

□

The proof above also proves the following extension of Lemma 4.3 from the setting of a closed target Y to the setting of a crowned surface Y.

LEMMA 4.4. *Let R_0 be the constant from Theorem 3.9. Let $f: X \to Y$ be a harmonic diffeomorphism from a punctured Riemann surface (possibly disconnected) to a crowned hyperbolic surface Y. Let $e(f)$ be the energy density with respect to the metric induced by the Hopf differential* Hopf(f). *Then, for any $R > R_0$,*

$$\int_{X \setminus \mathscr{P}_R} |e(f) - 2| \, dA \leqslant C |\chi(X)| e^{-R/2},$$

where C is a constant depending on $\chi(X)$, $\mathscr{P}_R$ is the Minsky's polygonal region and dA is the area measure induced by the Hopf differential Hopf(f).

4.2. Harmonic maps with varying domains

In this subsection, we shall prove a compactness result for harmonic diffeomorphisms with a fixed target but with varying domains. To start, we briefly explain the notion of convergence of harmonic maps with varying domains. For more detailed description of this notion, we refer to [33, §3 and §4]. Let $Y \in \mathcal{T}(S)$ be a fixed hyperbolic surface. Let $X_n \in \mathcal{T}(S)$ be an arbitrary divergent sequence of Riemann surfaces, and let Φ_n be the Hopf differential of the harmonic map $f_n: X_n \to Y$ homotopic to the change of marking. Let $R_m > 0$ be a sequence of divergent positive real numbers. Since the number of zeros of Φ_n is at most $2|\chi(S)|$, it follows that there exists a positive integer $k \leqslant 2|\chi(S)|$ such that, for sufficiently large m, the polygonal region $\mathscr{P}_{R_m}(\Phi_n)$ contains exactly k components, up to a subsequence of (X_n, Φ_n). Choose a zero for each of the components of $\mathscr{P}_{R_m}(\Phi_n)$. Let $\mathfrak{p}_n = \{p_{1,n}, ..., p_{k,n}\}$ be the choice of zeros of Φ_n. Consider the family of pointed singular flat surfaces $(\mathscr{P}_{R_m}(\Phi_n); \mathfrak{p}_n)$. By Lemma 4.1, these pointed surfaces have bounded injective radius and bounded diameter, hence (sub)converge to some pointed singular flat surface $(Z_m; \mathfrak{q})$ in the Gromov–Hausdorff topology. Letting $m \to \infty$ and applying a diagonal argument, we get a nested sequence of singular flat surfaces

$$(Z_1; \mathfrak{p}) \subsetneq (Z_2; \mathfrak{p}) \subsetneq ... \subsetneq (Z_m; \mathfrak{p}) \subsetneq ...$$

such that there exists a subsequence of $(\Phi_n; \mathfrak{p}_n)$ converging to the pointed singular flat surface $(\bigcup_{m \geqslant 1} Z_m; \mathfrak{p})$. For simplicity, we still denote this subsequence by $(\Phi_n; \mathfrak{p}_n)$. Let X be the Riemann surface underlying $\bigcup_{m \geqslant 1} Z_m$. Then, there exists a family of quasiconformal embeddings $\iota_{m,n}: \mathscr{P}_{R_m}(\Phi_n) \to X$ with quasiconformal constant uniformly converging to 1 as $n \to 1$, whose images exhaust X, that is,

$$\bigcup_{m \geqslant 1} \lim_{n \to \infty} \iota_{m,n}(\mathscr{P}_{R_m}(\Phi_n)) = X.$$

Applying a standard energy estimate (see, e.g., [12, Lemma 3], [98, Proposition 3.1] or [81, Proposition 1.3]), we see that, up to a subsequence if necessary, the sequence $f_n\colon (X_n;\mathfrak{p}_n)\to Y$ converges to a harmonic map $f\colon (X;\mathfrak{p})\to Y$ with Hopf differential Φ in the following sense. Take an arbitrary compact exhaustion $\{\mathcal{K}_j\}$ of X. For each j, and for sufficiently large m, the sequence of composition maps $f_n\circ(\iota_{m,n})^{-1}|_{\mathcal{K}_j}\colon\mathcal{K}_j\to Y$ converges to $f|_{\mathcal{K}_j}\colon\mathcal{K}_j\to Y$ uniformly. (We also consider the situation where $Y_n\to Y$. Let $\xi_n\colon Y_n\to Y$ be a diffeomorphism with Lipschitz constant converging to 1 as $n\to\infty$. Consider the harmonic diffeomorphism $h_n\colon (X_n;\mathfrak{p}_n)\to Y_n$. An analogous argument shows that $h_n\colon (X_n;\mathfrak{p}_n)\to Y_n$ converges to a harmonic map $h\colon (X;\mathfrak{p})\to Y$ with Hopf differential Φ: for each j, and for sufficiently large m, we have that the sequence of composition maps $\xi_n\circ h_n\circ(\iota_{m,n})^{-1}|_{\mathcal{K}_j}\colon\mathcal{K}_j\to Y$ converges to $f|_{\mathcal{K}_j}\colon\mathcal{K}_j\to Y$ uniformly.)

In the rest of this paper, for the sake of simplicity, we identify $U\subset X$ with its images $\iota_{m,n}^{-1}(U)\subset X_n$ without mentioning the map $\iota_{m,n}^{-1}$.

We next state a result regarding limits of sequences of harmonic maps and the limit of the associated Hopf differentials. In this direction, we define our notion of limit in this setting.

Definition 4.5. Let $X_n, Y_n, Y\in\mathcal{T}(S)$. Let $f_n\colon X_n\to Y_n$ be a harmonic diffeomorphism with Hopf differential Φ_n. Let $f_\infty\colon X_\infty\to Y$ be a harmonic map (not necessarily surjective) from a (possibly disconnected) punctured Riemann surface X_∞ to Y with Hopf differential Φ_∞. We say that $f_n\colon X_n\to Y_n$ *converges to* $f\colon X_\infty\to Y$ if the following conditions hold:

(i) each component of X_∞ equipped with the $|\Phi_\infty|$-metric is a Gromov–Hausdorff limit of X_n under the $|\Phi_n|$-metric for some choice of base points $p_n\in\mathscr{Z}_n$, where $\mathscr{Z}_n$ is the set of zeros of Φ_n;

(ii) the surface X_∞ equipped with the $|\Phi_\infty|$-metric contains all pointed Gromov–Hausdorff limits of (X_n, p_n) under the $|\Phi_n|$-metric with base point $p_n\in\mathscr{Z}_n$;

(iii) on each Gromov–Hausdorff component U_∞ of X_∞, we have that the harmonic map f_∞ is the limit of the harmonic maps f_n on a domain $U_n\subset X_n$, where U_n converges to U_∞.

LEMMA 4.6. (Compactness) *Let $Y\in\mathcal{T}(S)$ be a hyperbolic surface. Let $X_n\in\mathcal{T}(S)$ be an arbitrary divergent sequence of Riemann surfaces with $f_n\colon X_n\to Y$ being the corresponding harmonic diffeomorphism. Then, there exist a chain-recurrent geodesic lamination λ on Y, and a subsequence $f_{n_m}\colon X_n\to Y$ which converges to a surjective harmonic diffeomorphism $f\colon X\to Y\setminus\lambda$ from some (possibly disconnected) punctured surface X. Moreover, the following conditions hold*:

- *the quadratic differential* Hopf(f) *has a pole of order at least* 2 *at each puncture*;

- *we have*

$$\lambda = \lim_{R\to\infty} \lim_{m\to\infty} \overline{f_n(X_{n_m} \setminus \mathscr{P}_R(\mathrm{Hopf}(f_{n_m})))}.$$

Proof. We continue using the notation introduced in the beginning of this subsection. In particular, X is the Riemann surface underlying the singular flat metric

$$\bigcup_{m\geqslant 1} Z_m.$$

Notice that on each connected component of X, the limiting harmonic map f is a harmonic diffeomorphism onto its image ([98, proof of Proposition 3.4], following [78]). By Theorem 3.7, for sufficiently large n, the image $f_n(X_n \setminus \mathscr{P}_{R_n}(\Phi_n))$ is contained in some ε_n-neighbourhood of the geodesic lamination corresponding to the horizontal measured foliation of Φ_n, with $\varepsilon_n \to 0$ as $n\to\infty$. This implies that $f_n(X_n \setminus \mathscr{P}_{R_n}(\Phi_n))$ converges to some geodesic lamination λ as $n\to\infty$ in the Hausdorff topology. Recall that every measured lamination is chain-recurrent. Combining with the fact that the Hausdorff limit of a sequence of chain-recurrent laminations is again chain-recurrent ([90, Proposition 6.2]), we see that λ is a chain-recurrent.

We claim that $f(X) \subset Y \setminus \lambda$. Suppose to the contrary that there is some $x\in X$ such that $f(x)\in\lambda$. Then, there is a neighbourhood V of $f(x)$ with $V\subset f(X)$. As $f_n: X_n \to Y$ converges to $f: X\to Y$, it follows that there exist small neighbourhoods $U_n\subset X_n$, with $f_n(U_n)\subset V$ and also with U_n approximating some fixed region in X, and hence at a uniformly bounded distance from the zeros of Φ_n. On the other hand, the assumption that λ is the Hausdorff limit of $f_n(X_n \setminus \mathscr{P}_{R_n})$ implies that there exists a sequence of points $p_n \in X_n \setminus \mathscr{P}_{R_n}$, whose distance to the zeros of Φ_n diverges, such that $f_n(p_n) \to f(x) \in \lambda$. In particular, $f_n(p_n) \in f_n(U_n)$ but $p_n \notin U_n$. This contradicts the fact that f_n is a homeomorphism.

Recall that (from the second paragraph of the proof) $f_n(X_n \setminus \mathscr{P}_{R_n}(\Phi_n))$ converges to λ as $n\to\infty$. Combined with the fact that $f_n(X_n)\equiv Y$, this implies that $f(X)$, being the limit of $f_n(\mathscr{P}_{R_n}(\Phi_n))$ as $n\to\infty$, contains $Y\setminus\lambda$. On the other hand, the claim in the previous paragraph implies that $f(X)\subset Y\setminus\lambda$. Hence, we have $f(X)=Y\setminus\lambda$.

Finally, we show that $f: X\to Y$ is globally injective. Recall that for each component X^i of X, the restriction $f|_{X^i}$ is injective. In particular, f is open. To prove that $f: X\to Y$ is globally injective, it suffices to show that, for $i\neq j$, we have $f(X^i)\cap f(X^j)=\varnothing$. Suppose to the contrary that there exist $x^i\in X^i$ and $x^j\in X^j$ such that $f(x^i)=f(x^j)$. Then, there exist disjoint neighbourhoods U^i of x^i and U^j of x^j such that $f(U^i)=f(U^j)$. It follows that, for n sufficiently large, $f_n(U^i)\cap f_n(U^j)\neq\varnothing$. Again, this contradicts the fact that f_n is a homeomorphism for every n. □

5. The generalized Jenkins–Serrin problem

In this section, we consider minimal surfaces in $M\times T$, where M is a hyperbolic surface and T is a tree satisfying some conditions. The story begins with Scherk's example ([76], [44]), which is a minimal graph over a square in $\mathbb{R}^2$ with boundary values $\pm\infty$ alternately, known as "the Dirichlet problem with infinite boundary values". This was generalized to minimal graphs over $2n$-gons in $\mathbb{R}^2$ in [44] and over ideal $2n$-gons in $\mathbb{H}^2$ in [66]. (In [44] and [66], the domains also allow strictly convex arcs as part of the boundary.) In these cases, the surface M is a polygon with an even number of edges and the tree T is simply the real axis $\mathbb{R}$. For our purpose, we need to consider the case where M is the universal cover of a "hyperbolic crowned surface" and T is a tree dual to some measured foliation. In particular, by extending to trees (which may not admit a folding to a real line), we extend the scope of the results to include the case where M is a hyperbolic ideal polygon with an odd number of edges. This section concerns the uniqueness problem of minimal graphs (see Theorem 5.7). The existence problem will be addressed in Appendix A (see Theorem A.1).

5.1. Minimal graphs over domains in $\mathbb{H}^2$

In this subsection, we collect some results about the minimal graphs over domains in $\mathbb{H}^2$. For more details, we refer to [66]. Consider the unit disk model of $\mathbb{H}^2$. Denote by (x_1,x_2,x_3) the coordinates on the product $\mathbb{H}^2\times\mathbb{R}$. The metric on $\mathbb{H}^2\times\mathbb{R}$ is

$$d\sigma^2=\frac{dx_1^2+dx_2^2}{F}+dx_3^2,$$

where

$$F=\left(\frac{1-x_1^2-x_2^2}{2}\right)^2.$$

Let $D\subset\mathbb{H}^2$ be a hyperbolic domain. The graph of a function $u\colon D\to\mathbb{R}$ is *minimal* if and only if satisfies the *minimal surface equation*:

$$\operatorname{div}\left(\frac{\nabla u}{\tau_u}\right)=0, \tag{5.1}$$

where

$$\tau_u=\sqrt{1+|\nabla u|^2},$$

and ∇u and div are the gradient and divergence with respect to $\mathbb{H}^2$.

Let u be a solution of the minimal surface equation. It is clear that the differential

$$du^*:=\frac{F(u_1\,dx_2-u_2\,dx_1)}{\tau_u} \tag{5.2}$$

is closed on D, where

$$u_i := \frac{\partial u}{\partial x_i}.$$

Locally, we may then define a function u^* on D, uniquely up to an additive constant, the *conjugate function* of u. Geometrically, we can interpret du^* as follows. Let $U \subset D$ be a subdomain and $\alpha \subset \partial U$ a boundary arc with arc length parametrization s such that the domain is on the left. Then,

$$\int_\alpha du^* = \int_\alpha \left\langle \frac{\nabla u}{\tau_u}, \nu \right\rangle ds,$$

where ν is the outward unit conormal to U along α. (The integral $\int_\alpha du^*$ is called the *flux* of u across α.) In particular,

$$\int_\alpha |du^*| \leqslant |\alpha|, \tag{5.3}$$

where $|\alpha|$ is the hyperbolic length of α.

Suppose $\gamma \subset \mathbb{H}^2$ is a geodesic segment. Consider the infinite strip bounded by the two geodesics which are orthogonal to γ and which pass through the endpoints of γ. For each $\varepsilon > 0$, there are two level curves contained in this strip each of which consists of points of distance ε to γ. Each of these two level curves is said to be an ε-translate of γ. We need the following estimate from [66, Lemma 1], which we state slightly differently here. Geometrically, this lemma says that the tangent planes near "divergence points" are almost vertical.

THEOREM 5.1. ([66, Lemma 1]; see also [19, flux theorem]) *Let $D \subset \mathbb{H}^2$ be a convex domain. Let $\alpha \subset \partial D$ be a compact geodesic arc. Let $\alpha_\varepsilon \subset D$ be an ε-translate of α. Let $u: D \to \mathbb{R}$ be a solution of the minimal surface equation* (5.1).

If $u|_\alpha = +\infty$, then the following properties hold:

(1) *$\langle \nabla u/\tau_u, \nu \rangle$ converges uniformly to 1 as $\varepsilon \to 0$, where ν is the unit field normal to α_ε and pointing toward α*;

(2)

$$\lim_{\varepsilon \to 0} \int_{\alpha_\varepsilon} du^* = |\alpha|,$$

where $|\alpha|$ is the hyperbolic length of α.

If $u|_\alpha = -\infty$, then the following properties hold:

(3) *$\langle \nabla u/\tau_u, \nu \rangle$ converges uniformly to -1 as $\varepsilon \to 0$, where ν is the unit field normal to α_ε and pointing toward α*;

(4)

$$\lim_{\varepsilon \to 0} \int_{\alpha_\varepsilon} du^* = -|\alpha|,$$

where $|\alpha|$ is the hyperbolic length of α.

5.2. Minimal graphs in $M \times T$

Given a measured foliation F on a crowned hyperbolic surface Y, the *critical graph* of F is the union of critical leaves that either connect singularities of F or leave all compacta of F. In particular, the critical graph does not include half-infinite but precompact critical leaves. (Compare Definition 2.7.)

Definition 5.2. (Admissible foliations) A measured foliation F on Y is said to be *admissible* if it satisfies the following properties.

(I) Each component in the complement of the critical graph of F is either (i) precompact, or (ii) a half-infinite cylinder foliated by closed leaves homotopic to a closed boundary geodesic, or (iii) an infinite strip of finite height foliated by homotopically nontrivial bi-infinite leaves such that each end of the strip either spirals around a closed boundary geodesic or is asymptotic to an ideal point of a crown end, or (iv) a half-plane foliated by bi-infinite leaves parallel to a single ideal geodesic boundary arc.

(II) Each closed boundary geodesic Y is either parallel to the closed leaves of one half-infinite cylinder component of F or is the (spiral) limit of (at least) one infinite strip component of F, in the sense that each leaf in the strip limits on the closed boundary geodesic.

(III) Each ideal geodesic boundary arc of Y is parallel to the bi-infinite leaves of one half-plane component of F.

(IV) Each ideal point of a crown end of Y is asymptotic to (at least) one half-infinite critical leaf.

Remark 5.3. An admissible measured foliation F on a crowned hyperbolic surface Y has finitely many precompact components, finitely many half-infinite cylinders (at most the number of closed boundary geodesics of Y), finitely many infinite strips (since strips are pairwise disjoint), and finitely many half-planes (the same number as that of ideal boundary geodesic arcs of crown ends). Accordingly, an admissible measured foliation has only a finite number of vertices/singularities, each of finite valence.

Definition 5.4. (Admissible dual trees) Let Y be a crowned hyperbolic surface. The (metric) tree dual to the lift of an admissible measured foliation on Y to the universal cover $\widetilde{Y}$ is said to be an *admissible dual tree.*

Remark 5.5. Note that the lift of a proper path to a boundary curve or a crown projects to a half-infinite path in the tree.

Let T be the dual tree of the lift of some admissible measured foliation F on Y. Let $\iota: \widetilde{Y} \to T$ be the projection map along leaves of $\widetilde{F}$. Consider the boundary behaviour of ι. If $p \in \widetilde{Y}$ approaches some point on a lift of an ideal boundary geodesic arc, then $\iota(p)$ goes to infinity along a half-infinite ray dual to the lift of the half-infinite plane of F given by item (III) of Definition 5.2. If p approaches some point on a lift of a closed boundary geodesic that is parallel to the closed leaves of a half-infinite cylinder of F, then $\iota(p)$ goes to infinity along a half-infinite ray dual to the lift of that half-infinite cylinder. If p approaches some point on a lift of a closed boundary geodesic that is spiralled towards by (finitely many) infinite strips of F, then $\iota(p)$ goes to infinity along a half-infinite ray dual to the lift of those strips. Here, distinct lifts of these strips induce adjoining finite segments in the tree which collect to form a single half-infinite ray; this ray is the image in the tree of the lift of a neighborhood of a finite subsegment of the boundary (compare Lemma 5.9 (iv)). Let $\partial_{\mathrm{hi}}T \subset \partial T$ be the subset of the Gromov boundary ∂T of T, which consists of ends determined by the three types of half-infinite rays mentioned above. In summary, the map ι induces a *partial boundary map*

$$\partial\iota\colon \partial_{\mathrm{ibg}}\widetilde{Y} \longrightarrow \partial_{\mathrm{hi}}T,$$

where $\partial_{\mathrm{ibg}}\widetilde{Y}$ is the union of the lift of closed boundary geodesics and ideal boundary geodesic arcs of Y. Note that any other admissible measured foliation F' equivalent to F differs from F by an isotopy followed by a sequence of Whitehead moves, that is, splitting or collapsing singularities of admissible measured foliations. Hence, the partial boundary map $\partial\iota$ is independent of the choice of F in its equivalence class. A partial boundary map $\partial_{\mathrm{ibg}}\widetilde{Y} \to \partial_{\mathrm{hi}}T$ is said to be an *admissible partial boundary map* if it is the partial boundary map of some projection map $\widetilde{Y} \to T$ along the leaves of an admissible measured foliation.

Remark 5.6. Here, the map ι is not defined on the Gromov boundary of $\widetilde{Y}$. For instance, it is not defined at any boundary cusp of $\widetilde{Y}$. In fact, for any sequence of points $p_i \in \widetilde{Y}$ converging to a boundary cusp bounded by two ideal geodesic boundary arcs $\gamma^{\pm}$, the accumulation set of the image sequence $\iota(p_i)$ could be the bi-infinite geodesic on T containing the two half-infinite edges corresponding to the two ideal geodesic boundary arcs $\gamma^{\pm}$. What we actually need (and describe) is the behavior of ι as one approaches an ideal geodesic boundary arc of $\widetilde{Y}$.

Our main result in this section is the following uniqueness result, which we will rely on in places in order to show the well-definedness of some limits of sequences.

THEOREM 5.7. *Let Y be a crowned hyperbolic surface. Let T be an admissible dual tree. Then, there exists at most one $\pi_1(Y)$-equivariant minimal graph in $\widetilde{Y} \times T$ over $\widetilde{Y}$ with a prescribed admissible partial boundary map.*

Here, a *minimal graph* in $\widetilde{Y}\times T$ over $\widetilde{Y}$ is a map $u:\widetilde{Y}\to T$ that satisfies the following properties:

- There is a discrete set $\{p_i\}\subset\widetilde{Y}$ of singular points such that for any $p\neq p_i$, there exists a neighbourhood $U\subset\widetilde{Y}$ of p, an interval $I\subset\mathbb{R}$, and an isometric embedding $j: I\to T$ such that: (i) $u(U)\subset j(I)$, and (ii) the composition map $j^{-1}\circ u: U\to I$ satisfies the *minimal surface equation* (5.1).
- At each singular point p_i, there exists a neighborhood U of p_i disjoint from any other singular point, a k-valence star V of finite total length ($3\leqslant k<\infty$), and an isometric embedding $j: V\to T$ that sends the center of V to $u(p_i)$ such that: (i) $u(U)\subset j(V)$, and (ii) the composition map $j^{-1}\circ u: U\to V$ is a projection map along leaves of a (singular) measured foliation on U whose leaf space is isometric to V.
- Moreover, the projection maps $(z,u(z))\mapsto z$ and $(z,u(z))\mapsto u(z)$, from the graph of u in $\widetilde{Y}\times T$ to either factor, are harmonic.

In light of the discussion in §3.3, such a minimal graph induces an (equivariant) conformal and harmonic product map from $\widetilde{X}$ to the product $\widetilde{Y}\times T$, where $\widetilde{X}$ is the conformal structure induced on the image

$$\{(p,u(p))\in\widetilde{Y}\times T\}.$$

We briefly describe the organization of the proof. Of course, we want to compare the minimal graphs of two maps from $\widetilde{Y}$ to T, say

$$u:\widetilde{Y}\longrightarrow T \quad\text{and}\quad v:\widetilde{Y}\longrightarrow T.$$

The proof is divided into two steps. In the first step, we prove that the distance function $\operatorname{dist}(u(\cdot),v(\cdot))$ is bounded and the supremum is realized at some point. The idea of this step is to fix a point $p\in\widetilde{Y}$ which is not a zero of any Hopf differential, and then, for any q, consider the distances

$$\mathbf{u}(q):=u(q)-u(p)\quad\text{and}\quad\mathbf{v}(q):=v(q)-v(p).$$

These distances are not well defined in a tree, but we are principally interested in a form

$$\widetilde{\Psi}=(\mathbf{u}-\mathbf{v})d(\mathbf{u}^*-\mathbf{v}^*),$$

and we find that this form is well-defined on a neighbourhood of $\widetilde{\partial Y}$ and descends to a neighborhood of ∂Y. Analyzing the level sets of $\operatorname{dist}(u(\cdot),v(\cdot))$ near ∂Y, we also find that $d\mathbf{u}^*-d\mathbf{v}^*$ is "nearly well defined" on simply connected domains near ∂Y (regardless of the number of ideal geodesics of each crown end), and on cylindrical domains near crown ends which have an even number of ideal geodesics, which will then turn out to be the only case which remains. Estimating these forms, and applying Stokes' theorem, we prove that $\mathrm{dist}(u(\cdot), v(\cdot))$ is bounded and the supremum is realizable. In the second step, we prove that $\mathrm{dist}(u(\cdot), v(\cdot))$ is identically zero, using the maximum principle and the fact that harmonic maps are conformal at zeros of Hopf differentials. This concludes the outline. We now describe the argument more precisely.

Remark 5.8. The existence problem about the equivariant minimal graphs in $\widetilde{Y}\times T$ will be addressed in the appendix (see Theorem A.1). As a direct consequence, we are able to parameterize harmonic diffeomorphisms from the complex plane to any ideal hyperbolic n-gon, using trees with n half-infinite prongs (and no vertices of valence one). Let P be an ideal hyperbolic n-gon. Let $\mathrm{ADT}(P)$ be the set of admissible dual trees of P. Then, there is a bijection between $\mathrm{ADT}(P)$ and the set of harmonic diffeomorphisms from $\mathbb{C}$ to P. From [64, Theorem 3.3] or [35, Proposition 3.5], it follows that $\mathrm{ADT}(P)$ is homeomorphic to $\mathbb{R}^{n-3}$. (Consider the admissible tree T_0 with only one vertex and n half-infinite prongs. Then, every other admissible dual tree of P is obtained from T_0 by replacing the vertex with a metric tree, a "metric expansion" of T_0. In other words, the set $\mathrm{ADT}(P)$ is the set of metric expansions of T_0.)

For $u:\widetilde{Y}\to T$ the equivariant map from $\widetilde{Y}$ to T, we let $\widetilde{X}$ be the Riemann surface underlying the graph $(z, u(z))$, and we let $\tilde{f}:\widetilde{X}\to\widetilde{Y}$ be the equivariant projection map $(z, u(z))\mapsto z$ from $\widetilde{X}$ to $\widetilde{Y}$, and $f: X\to Y$ be its descent.

Let $\{a_i, c_j\}$ be the set of punctures of X labeled in such a way that a_i corresponds to the closed geodesic boundary α_i of Y, while c_j corresponds to the crown end C_j of Y. Then, we have the following result.

LEMMA 5.9. (i) *Each puncture a_i is a pole of order* 2, *with residue of non-vanishing real part, of* $\mathrm{Hopf}(f)$ *and c_j is a pole of order k_j of* $\mathrm{Hopf}(f)$, *where $k_j\geqslant 3$ is the number of ideal geodesics contained in the boundary of the crown end C_j.*

(ii) *The tree T is the dual tree to the horizontal measured foliation of* $\mathrm{Hopf}(\tilde{f})$.

(iii) *Let γ be an arbitrary ideal geodesic in ∂Y. Then, for any lift $\widetilde{\gamma}$ and any compact subsegment $I\subset\widetilde{\gamma}$, there is a convex domain $U\subset\widetilde{Y}$ with $I\subset\partial U$ such that $u(U)$ is contained in a single half-infinite edge of T, and $u(p)\to\infty$ as $p\to I$.*

(iv) *Let α be an arbitrary closed geodesic loop in ∂Y. Then, for any lift $\tilde{\alpha}$ and any compact subsegment $I\subset\tilde{\alpha}$, there is a convex domain $U\subset\widetilde{Y}$ with $I\subset\partial U$ and a geodesic ray $r:[0,+\infty)\to T$ such that $u(U)$ is contained in the image of r and $u(p)\to\infty$ as $p\to I$.*

Here, the phrasing that $u(p)\to\infty$ as $p\to I$ means that $u(p)$ leaves all compact sets in the half-infinite (half-closed) path of T, as p approaches the segment I.

Proof. The first two items follow from [34] except for the statement that the residue at a_i has non-vanishing real part. Suppose that the residue at some a_i is purely imaginary then the half-infinite cylinder corresponding to a_i is vertical. Let ω_d be the core curve whose distance to the compact boundary of this half-infinite cylinder is exactly d. By Theorem 3.9, the length of image $f(\omega_d)$ under the harmonic map $f: X \to Y$ converges to zero as $d \to \infty$. This contradicts the fact that the length of non-trivial simple closed curves on Y have a uniform lower bound away from zero. Hence, the residue cannot be purely imaginary.

For the third item, let $U' \subset \widetilde{X}$ be the half-plane (cf. also [34]) corresponding to $\widetilde{\gamma}$. Then, $\widetilde{\gamma} \subset \partial \tilde{f}(U')$ and $u(\tilde{f}(U'))$ is contained in the half-infinite edge of T corresponding to $\widetilde{\gamma}$. We may choose U to be a convex domain of $\tilde{f}(U')$ with $I \subset \partial U$.

It remains to show the fourth item. By the first item of this lemma, we know that the puncture corresponding to α is a second order pole of Φ. (Here, from the first statement, the residue of this second order pole has non-vanishing real part. In the following argument, the non-vanishing of the real part of the residue will play no role; note that the assumption that α has positive length precludes the case of a purely imaginary residue.) This gives a half-infinite cylinder C in the flat metric $|\Phi|$ (which is horizontal if and only Φ has purely real residue at this puncture). Let $\widetilde{C}$ be a lift of C corresponding to $\tilde{\alpha}$. Then, $\tilde{f}(\widetilde{C})$ is a simply connected domain of $\widetilde{Y}$ with $\tilde{\alpha} \subset \partial \tilde{f}(\widetilde{C})$. Moreover, $u(\tilde{f}(\widetilde{C}))$ is a geodesic ray with $u(p) \to \infty$ as $p \in \tilde{f}(\widetilde{C})$ approaches $\tilde{\alpha}$. We may choose U to be a convex domain of $\tilde{f}(\widetilde{C})$ with $I \subset \partial U$. □

5.3. Two differentials

Recall that $u: \widetilde{Y} \to T$ and $v: \widetilde{Y} \to T$ are two $\pi_1(Y)$-equivariant minimal graphs in $\widetilde{Y} \times T$ with the same admissible partial boundary map.

Let $\widetilde{X}_u$ be the graph of $u: \widetilde{Y} \to T$ in $\widetilde{Y} \times T$, and $\tilde{f}_u: \widetilde{X}_u \to \widetilde{Y}$ the equivariant projection map which is harmonic. Then, T is dual to the horizontal measured foliation of $\mathrm{Hopf}(\tilde{f}_u)$. Let f_u be the projection map descended from $\tilde{f}_u$. Consider the map $v: \widetilde{Y} \to T$. Define $\widetilde{X}_v$ and $\tilde{f}_v$ similarly. Then, T is also dual to the horizontal measured foliation of $\mathrm{Hopf}(\tilde{f}_v)$. Let $X_u = \widetilde{X}_u / \pi_1(Y)$ and $X_v = \widetilde{X}_v / \pi_1(Y)$ be the quotient surfaces. Then, the horizontal measured foliations $\mathrm{Hor}(\mathrm{Hopf}(f_u))$ and $\mathrm{Hor}(\mathrm{Hopf}(f_v))$ of $\mathrm{Hopf}(f_u)$ and $\mathrm{Hopf}(f_v)$, respectively, are topologically equivalent. Let

$$F_u = f_u(\mathrm{Hor}(\mathrm{Hopf}(f_u))) \quad \text{and} \quad F_v = f_v(\mathrm{Hor}(\mathrm{Hopf}(f_v)))$$

be the associated measured foliations on Y. Since f_u and f_v are both homotopic to the identity, it follows that F_u and F_v differ by an isotopy and Whitehead moves. Let $\mathrm{Sing}(u)$

be the set of singular points of F_u, $\mathrm{Crit}(u)$ be the union of critical leaves of F_u and $\mathrm{Reg}(u)$ be the complement of $\mathrm{Crit}(u)\cup\mathrm{Sing}(u)$ in Y. Let $\widetilde{\mathrm{Reg}}(u)$, $\widetilde{\mathrm{Crit}}(u)$ and $\widetilde{\mathrm{Sing}}(u)$ be the lifts to $\widetilde{Y}$ of $\mathrm{Reg}(u)$, $\mathrm{Crit}(u)$ and $\mathrm{Sing}(u)$, respectively. Then, the following properties hold:

- $\widetilde{\mathrm{Reg}}(u)$ is the subset of points $p\in\widetilde{Y}$ such that $u(p)$ is not a vertex of T;
- $\widetilde{\mathrm{Crit}}(u)$ is the subset of points $p\in\widetilde{Y}$ such that $u(p)$ is a vertex of T but p admits a neighbourhood whose image under u is a geodesic segment;
- $\widetilde{\mathrm{Sing}}(u)$ is the subset of points $p\in\widetilde{Y}$ such that $u(p)$ is vertex and p admits a neighbourhood whose image under u is a star with $m\geqslant 3$ edges/prongs.

Let $\mathrm{Sing}(v)$, $\mathrm{Crit}(v)$, $\mathrm{Reg}(v)$, $\widetilde{\mathrm{Sing}}(v)$, $\widetilde{\mathrm{Crit}}(v)$ and $\widetilde{\mathrm{Reg}}(v)$ be similarly defined.

We now begin our analysis of how the horizontal measured foliations F_u and F_v align, and the implications for the maps u and v.

LEMMA 5.10. *Let $p\in\widetilde{\mathrm{Reg}}(u)\cap\widetilde{\mathrm{Reg}}(v)$. Then, there exists a neighbourhood $U\subset\widetilde{Y}$ of p such that the convex hull of $u(U)\cup v(U)$ is a geodesic.*

Proof. Let U be a neighbourhood of p such that $u(U)$ and $v(U)$ are both geodesic segments. Let $\mathrm{hull}(U)$ be the convex hull of $u(U)\cup v(U)$ in T. If $\mathrm{hull}(U)$ is a geodesic segment, then we are done. Otherwise, the assumption that neither $u(p)$ nor $v(p)$ is a vertex of T implies that there exists a subdomain $U'\ni p$ such that $u(U')\cup v(U')$ avoids the (discrete set of) points of $\mathrm{hull}(U)$ which have valence at least three. It then follows that the convex hull of $u(U')\cup v(U')$ is a geodesic segment in T. □

Consider the minimal graphs $u\colon\widetilde{Y}\to T$ and $v\colon\widetilde{Y}\to T$. If there exists a folding $\xi\colon T\to\mathbb{R}$ (cf. [25] or [63]), then we can compare u and v by considering the difference $\xi\circ u-\xi\circ v$. Theorem 5.7 then follows directly from the argument in the proof of step 6 (uniqueness) of [66, Theorem 3]. But such a folding does not exist in general. Nevertheless, the observation below (Lemma 5.11) allows us to get past this obstruction.

Let $p\in\widetilde{\mathrm{Reg}}(u)\cap\widetilde{\mathrm{Reg}}(v)$. Let $U\subset\widetilde{Y}$ be a neighbourhood of p such that the convex hull of $u(U)\cup v(U)$ is a geodesic, which is denoted by $\mathbf{L}$ (Lemma 5.10). We notice that this geodesic admits two orientations; we pick one of these. Now, let us define two functions $\mathbf{u}$ and $\mathbf{v}$ on U as follows. For each $q\in U$, let

$$\mathbf{u}(q):=u(q)-u(p)\quad\text{and}\quad\mathbf{v}(q):=v(q)-v(p)$$

be the oriented distance from $u(p)$ to $u(q)$ and from $v(p)$ to $v(q)$, respectively; here, we may measure distance along the geodesic we have constructed. Let $\mathbf{u}^*$, $\mathbf{v}^*$, $d\mathbf{u}^*$ and $d\mathbf{v}^*$ be defined as in (5.2). Notice that both $\mathbf{u}-\mathbf{v}$ and $d(\mathbf{u}^*-\mathbf{v}^*)$ are independent on the particular choice of $p\in U$. If we reverse the orientation of the geodesic $\mathbf{L}$, then $\mathbf{u}-\mathbf{v}$ differs by a negative sign, so does $d(\mathbf{u}^*-\mathbf{v}^*)$. This implies that the differential

$$(\mathbf{u}-\mathbf{v})\,d(\mathbf{u}^*-\mathbf{v}^*)$$

is independent on the choice of the orientation of $\mathbf{L}$. Therefore, we obtain a well defined differential $\widetilde{\Psi}$ on

$$\widetilde{\mathrm{Reg}}(u)\cap\widetilde{\mathrm{Reg}}(v)=\widetilde{Y}\setminus\big(u^{-1}(u(\widetilde{\mathrm{Sing}}(u)))\cup v^{-1}(v(\widetilde{\mathrm{Sing}}(v)))\big),$$

the complementary region of the union of the preimages of the vertices of T by u and v. Locally, the differential $\widetilde{\Psi}$ can be represented as $(\mathbf{u}-\mathbf{v})\,d(\mathbf{u}^*-\mathbf{v}^*)$.

Notice that $\widetilde{\Psi}$ is not closed in general. In fact, we have

$$\begin{aligned} d\widetilde{\Psi} &= \left[(\mathbf{u}_1-\mathbf{v}_1)\left(\frac{\mathbf{u}_1}{\tau_{\mathbf{u}}}-\frac{\mathbf{v}_1}{\tau_{\mathbf{v}}}\right)+(\mathbf{u}_2-\mathbf{v}_2)\left(\frac{\mathbf{u}_2}{\tau_{\mathbf{u}}}-\frac{\mathbf{v}_2}{\tau_{\mathbf{v}}}\right)\right]dx_1\,dx_2 \\ &= \left(\frac{\tau_{\mathbf{u}}+\tau_{\mathbf{v}}}{2}\right)\left[\left(\frac{\mathbf{u}_1}{\tau_{\mathbf{u}}}-\frac{\mathbf{v}_1}{\tau_{\mathbf{v}}}\right)^2+\left(\frac{\mathbf{u}_2}{\tau_{\mathbf{u}}}-\frac{\mathbf{v}_2}{\tau_{\mathbf{v}}}\right)^2+\frac{1}{F}\left(\frac{1}{\tau_{\mathbf{u}}}-\frac{1}{\tau_{\mathbf{v}}}\right)^2\right]dx_1\,dx_2 \end{aligned} \tag{5.4}$$

where $\mathbf{u}_i=\partial\mathbf{u}/\partial x_i$, $\mathbf{v}_i=\partial\mathbf{v}/\partial x_i$, $\tau_{\mathbf{u}}$ and $\tau_{\mathbf{v}}$ are defined in (5.1); the second identity is taken from [66, p. 280].

We first extend the domain of definition of $\widetilde{\Psi}$ to the lift of a cylindrical neighbourhood of ∂Y.

LEMMA 5.11. *Let C be a closed geodesic loop boundary component or crown end of Y. There exists a cylindrical neighbourhood U of C such that the following holds.*

(i) *The differential $\widetilde{\Psi}$ is well defined on the lift of U to the universal cover and descends to a differential Ψ on U.*

(ii) *For any simple arc η of U which cuts U into a simply connected domain, the restriction of u and v to $U\setminus\eta$ are minimal graphs in $\mathbb{H}^2\times\mathbb{R}$.*

Proof. Notice that Y may have geodesic boundary components or crown ends. To prove the lemma, it suffices to find a cylindrical neighbourhood for each boundary component which satisfies the two mentioned properties.

Let α be an arbitrary geodesic boundary loop of Y. Let $I\subset\tilde{\alpha}$ be a segment of a lift $\tilde{\alpha}$ of α to $\widetilde{Y}$ with $\ell(I)>\ell(\alpha)$. Then, by item (iv) in Lemma 5.9, we see that there exists a convex neighbourhood $U_I\subset\widetilde{Y}$ of I with $I\subset\partial U_I$ and a geodesic ray $r:[0,+\infty)\to T$ such that the following properties hold.

- both $u(U_I)$ and $v(U_I)$ are contained in the image of r;
- $u(p)\to\infty$ and $v(p)\to\infty$, as $p\to I$.

By the definition of $\widetilde{\Psi}$, we see that $\widetilde{\Psi}$ is well defined on any point in U_I. Any cylindrical subdomain of the quotient of U_I, which admits α as a boundary component, satisfies the conditions (i) and (ii).

Now, we move on to consider the case of crown ends. Let C be a crown end with ideal geodesic boundary arcs $\gamma_1,...,\gamma_k$ labelled cyclically. Let $\widetilde{X}_u$ and $\widetilde{X}_v$ be, respectively, the minimal graphs of $u:\widetilde{Y}\to T$ and $v:\widetilde{Y}\to T$. Let $f_u:\widetilde{X}_u\to\widetilde{Y}$ and $f_v:\widetilde{X}_u\to\widetilde{Y}$ be the harmonic projection maps described at the outset of this subsection. Consider the horizontal foliations $\mathrm{Hor}(\mathrm{Hopf}(f_u))$ and $\mathrm{Hor}(\mathrm{Hopf}(f_v))$ of the Hopf differentials of f_u and f_v, respectively. Recall that $\mathrm{Crit}(u)\subset Y$ is the image under f_u of the critical leaves of $\mathrm{Hor}(\mathrm{Hopf}(f_u))$, which is also the preimage of the vertices of T under u. Of course, $\mathrm{Crit}(v)$ is defined similarly. For the cusp P_i bounded by γ_i and γ_{i+1}, there exists a cusp neighbourhood W_i such that $W_i\cap(\mathrm{Crit}(u)\cup\mathrm{Crit}(v))$ consists of half-infinite simple arcs approaching the cusp P_i. Observe that these arcs are contained in the interior of W_i. We may take smaller cusp neighbourhoods so that for any $i\neq j$, we have that W_i and W_j are disjoint. We may then take U to be a cylindrical neighbourhood of the crown C such that the following conditions hold:

(a) $\partial U\cap\mathrm{int}(W_i)$ is connected and non-empty;

(b) $(\mathrm{Crit}(u)\cap U)\subset\bigcup_i W_i$ and $(\mathrm{Crit}(v)\cap U)\subset\bigcup_i W_i$.

In particular, this proves item (ii). It remains to prove (i). Let $\widetilde{W_i}$ be a connected component of the lift of W_i to the universal cover $\widetilde{Y}$. Then, for each W_i, we have that the images $u(W_i)=v(W_i)$, which is isometric to the real line $\mathbb{R}$ (by applying Lemma 5.9 (iii) twice). By the definition of $\widetilde{\Psi}$, we see that it is well defined on $\widetilde{W_i}$, and so $\tilde{\Psi}$ descends to a differential Ψ on W_i.

Consider the complementary components of $\mathrm{Crit}(u)\cup\mathrm{Crit}(v)$ in U. For each γ_i, there exists exactly one of these components, say R_i, which contains γ_i in the boundary. Let $\widetilde{R}_i$ be a connected component of the lift of R_i to the universal cover $\widetilde{Y}$. Let $\widetilde{\gamma}_i$ be a lift of γ_i which is contained in $\partial\widetilde{R}_i$. Let e_i be the half-infinite edge, corresponding to the leaves of $\mathrm{Hor}(\mathrm{Hopf}(f_u))$, or equivalently $\mathrm{Hor}(\mathrm{Hopf}(f_v))$, in the half-plane parallel to $\widetilde{\gamma}_i$, of T (here, the finite endpoint of e_i corresponds to the critical leaf of $\mathrm{Hor}(\mathrm{Hopf}(f_u))$, or equivalently $\mathrm{Hor}(\mathrm{Hopf}(f_v))$, bounding the half-plane parallel to $\widetilde{\gamma}_i$). Then, both $u(\widetilde{R}_i)$ and $v(\widetilde{R}_i)$ are contained in e_i. By the definition of $\widetilde{\Psi}$, we see that it is well defined on $\widetilde{R}_i$, which descends to a differential Ψ on R_i. Since $U=\bigcup_i(R_i\cup W_i)$, it follows that Ψ is well defined on U, and that $\widetilde{\Psi}$ is well defined on $\widetilde{U}$. This finishes the proof. □

If a crown end C consists of an even number of ideal geodesic arcs $\gamma_1,...,\gamma_{2n}$, then we can say a bit more about u and v as follows. Let $\mathrm{Hor}(\mathrm{Hopf}(f_u))$ and $\mathrm{Hor}(\mathrm{Hopf}(f_v))$ be as defined in the above proof (see also the beginning of §5.3). Let U and R_i also be as defined in the above proof. Then, the restriction to U of both $F_u:=f_u(\mathrm{Hor}(\mathrm{Hopf}(f_u)))$ and $F_v:=f_v(\mathrm{Hor}(\mathrm{Hopf}(f_v)))$ are orientable. We choose one orientation which induces the same orientation on $R_i\subset U$. Each choice of orientation yields two 1-forms ω_u and ω_v on U, defined by $F_u|_U$ and $F_v|_U$, respectively. For each cusp P_i of C, recalling §2.8, let $\mathrm{Strip}_i(u)$ (resp. $\mathrm{Strip}_i(v)$) be the restriction to the cusp region W_i (mentioned in the proof of Lemma 5.11) of the union of those (possibly degenerate) closed infinite-strips of F_u (resp. F_v) that approach P_i. The heights of $\mathrm{Strip}_i(u)$ and $\mathrm{Strip}_i(v)$ are determined by T, hence are equal. Let $h_i \geqslant 0$ be the height. Then, for any oriented simple closed curve α in U, we have

$$\int_\alpha \omega_u = \int_\alpha \omega_v = a \sum_{i=1}^{2n} (-1)^i h_i, \tag{5.5}$$

where $a \in \{1, -1\}$ depends on the orientation we select for the foliations on U as well as the orientation of α. In other words, both u and v induce functions from U to $\mathbb{R}/(h\mathbb{Z})$, where $h = \sum_{i=1}^{2n} (-1)^i h_i$. In particular, this implies that both $u-v$ and $du^* - dv^*$ are well defined on U. We summarize the discussion below.

LEMMA 5.12. + *Suppose that C is a crown end of Y which consists of an even number of ideal geodesic arcs. Let U be the cylindrical neighbourhood of C obtained from Lemma 5.11. Let F_u and F_v be defined as above. Then, there are compatible choices of orientations of the restriction $F_u|_U$ and $F_v|_U$ so that both $u-v$ and $du^* - dv^*$ are well defined on U.*

For any crown end with k ideal geodesic arcs $\gamma_1, ..., \gamma_k$, where k is not necessarily an even number, we have the following more general result. We continue with the notation as above. Recall that h_i is the height of the strip of $F_u|_U$ approaching the cusp P_i, which is also the height of the strip of $F_v|_U$ approaching the cusp P_i. Consider the measured foliations F_u. Let $\mathrm{HP}_i(u)$ be the half-plane corresponding to the ideal geodesic arc γ_i. Consider the half-infinite strip $\mathrm{Strip}_i(u)$ of F_u approaching P_i (constructed in the paragraph above (5.5)). Let $\mathbf{G}$ be the leaf space of $F_u|_{\bigcup_i (\mathrm{HP}_i(u) \cup \mathrm{Strip}_i(u))}$, i.e. the restriction of F_u to $\bigcup_i (\mathrm{HP}_i \cup \mathrm{Strip}_i(u))$. (Informally, then, the graph $\mathbf{G}$ is a collection of half-infinite prongs attached to a circle.) Let E_i be the half-infinite edge of $\mathbf{G}$ dual to $\mathrm{HP}_i(u)$. Let Q_i be the vertex of E_i, and $\overline{Q_i Q_{i+1}}$ be the finite edge of $\mathbf{G}$ dual to $\mathrm{Strip}_i(u)$. [Here, our notation allows $\overline{Q_i Q_{i+1}}$ to be degenerate, i.e. of zero length, so that some successive such edges $\overline{Q_i Q_{i+1}}$ might coincide (when all the strips incident to some successive cusps are degenerate).] Then, the orientation of Y induces a cyclic order of all edges of $\mathbf{G}$ as follows:

$$E_1, \quad \overline{Q_1 Q_2}, \quad E_2, \quad \overline{Q_2 Q_3}, \quad ..., \quad E_k, \quad \overline{Q_k Q_1}, \quad E_1. \tag{5.6}$$

In particular, the orientation of Y induces not only a cyclic order of the half-infinite edges corresponding to the successive finite edges of the graph $\mathbf{G}$ but also a cyclic order of edges incident at any common vertex of $\mathbf{G}$. Moreover, the map $u: \widetilde{Y} \to T$ descends to a map $u_U: U \to \mathbf{G}$. Considering the foliation F_v similarly, we get a similarly defined map $v_U: U \to \mathbf{G}$. In summary, we have the following result.

LEMMA 5.13. *Let C be a crown end with k ideal geodesic arcs $\gamma_1, ..., \gamma_k$. Let $\mathbf{G}$ be the graph defined as above. Then, u and v descend to maps $u_U: U \to \mathbf{G}$ and $v_U: U \to \mathbf{G}$, respectively.*

5.4. Boundedness of $\mathrm{dist}(u(\cdot), v(\cdot))$

We next apply the structure theory of the previous subsection to find that, for the Jenkins–Serrin maps we are studying, any pair of them have their maximum distance realized at (the lift of) an interior point of Y. Here, we see the importance of ensuring that Ψ is well defined in Lemma 5.11, as we are allowed to then use Stokes theorem, and especially the sign of the expression for $d\Psi$ in (5.4) in our computations. This subsection is devoted to a proof of the following lemma.

LEMMA 5.14. *Let $u: \widetilde{Y} \to T$ and $v: \widetilde{Y} \to T$ be two minimal graphs sharing the same admissible partial boundary map. Then, there exists a point $p \in \widetilde{Y}$ which realizes the supremum of the distance function on $\widetilde{Y}$, i.e.*

$$\mathrm{dist}(u(p), v(p)) = \sup_{q \in \widetilde{Y}} \mathrm{dist}(u(q), v(q)).$$

Proof. Consider the set U which is the union of the cylinder neighbourhoods of ∂Y obtained in Lemma 5.11. Set $K := Y \setminus U$. Then, K is a compact subset. Notice that the distance function $\mathrm{dist}(u(\cdot), v(\cdot))$ descends to Y. Suppose to the contrary that the conclusion (of the lemma) does not hold. Then, we have

$$\max_{p \in K} \mathrm{dist}(u(p), v(p)) < \sup_{p \in Y} \mathrm{dist}(u(p), v(p)). \tag{5.7}$$

Let M be an arbitrary constant such that

$$\max_{p \in K} \mathrm{dist}(u(p), v(p)) < M < \sup_{p \in Y} \mathrm{dist}(u(p), v(p)).$$

In particular, $M > 0$. Consider a component Ω of

$$\{p \in Y : \mathrm{dist}(u(p), v(p)) > M\}.$$

We have $\Omega \subset U$, as the points in the complement of U in Y have images at distance less than M. In the following, we shall show that $\mathrm{dist}(u(p), v(p)) \equiv M$ on Ω. The arbitrariness of M then implies that

$$\max_{p \in K} \mathrm{dist}(u(p), v(p)) = \sup_{p \in Y} \mathrm{dist}(u(p), v(p)),$$

which contradicts (5.7).

Consider the topology of Ω. There are three possibilities:

(i) Ω is simply connected and $\bar{\Omega}$ is contained in the interior of U;

(ii) Ω is simply connected but $\bar{\Omega}$ is not contained in the interior of U;

(iii) Ω is multiconnected.

Case (i): Ω is simply connected and $\bar{\Omega}$ is contained in the interior of U.

Then, we may take u and v as $\mathbb{R}$-valued functions with signs chosen so that $u-v=M$ on $\partial\Omega$. Thus, as both du^* and dv^* are locally well-defined closed differentials, we have

$$\int_{\partial\Omega} \Psi = M \int_{\partial\Omega} (du^* - dv^*) = 0.$$

Combining (5.4) and the Stokes theorem, we see that $\nabla u = \nabla v$ on Ω. Hence, $u-v$ is a constant function over Ω, which is exactly M since $u-v=M$ on $\partial\Omega$. (Note that the proof here relies on the assumption that the distance function is identically M on $\partial\Omega$ and avoids the value M in Ω, but does not rely on the specific assumption that $\operatorname{dist}(u(p), v(p)) > M$ for $p \in \Omega$. So the conclusion still holds if $\operatorname{dist}(u(p), v(p)) < M$ for $p \in \Omega$ and $\operatorname{dist}(u(p), v(p)) = M$ for $p \in \partial\Omega$. This observation will be used in case (iii-b).)

Case (ii): Ω is simply connected but $\bar{\Omega}$ is not contained in the interior of U.

We approximate Ω by a domain $\Omega^{\delta,\varepsilon}$ which is slightly separated from ∂Y, and we then prove that the error used in the approximation is negligible. Notice that any consecutive pair (γ_i, γ_{i+1}) of ideal geodesics in the crown end $\mathcal{C}$ are asymptotic, and hence determines a point at infinity which we denoted by P_i. Let δ be a small positive constant to be determined. Let F_i^δ be a neighbourhood of P_i which is bounded by a segment in γ_i, a segment in γ_{i+1}, and a horocyclic arc centered at P_i with length δ. Let

$$\gamma_i^\delta := \gamma_i \cap \partial\biggl(\Omega \setminus \biggl(\bigcup_i F_i^\delta\biggr)\biggr).$$

Let ε be another small constant. Let $\Omega^{\delta,\varepsilon} \subset \Omega \setminus (\bigcup_i F_i^\delta)$ be the subsurface in Ω consisting of points whose distance to the boundary of Y is at least ε. The boundary $\partial\Omega^{\delta,\varepsilon}$ consists of arcs $\gamma_i^{\delta,\varepsilon}$ corresponding to γ_i, arcs $h_i^{\delta,\varepsilon}$ contained in the horocycle boundary of F_i^δ, and a subarc $\omega^{\delta,\varepsilon}$ of $\partial\Omega \cap U$. In particular,

$$|h_i^{\delta,\varepsilon}| < \delta \tag{5.8}$$

for each $i = 1, 2, ..., k$. Since both u and v are well-defined $\mathbb{R}$-valued functions over Ω,

$$\int_{\partial\Omega\cap U} \Psi = M \int_{\partial\Omega\cap U} (du^* - dv^*) \quad \text{and} \quad \int_{\omega^{\delta,\varepsilon}} \Psi = M \int_{\omega^{\delta,\varepsilon}} (du^* - dv^*). \tag{5.9}$$

Combining with the fact that both du^* and dv^* are closed differentials on Ω, we see that

$$\begin{aligned} \int_{\omega^{\delta,\varepsilon}} (du^* - dv^*) + \sum_{i=1}^k \int_{\gamma_i^{\delta,\varepsilon}} (du^* - dv^*) + \sum_{i=1}^k \int_{h_i^{\delta,\varepsilon}} (du^* - dv^*) &= \int_{\Omega^{\delta,\varepsilon}} d(du^* - dv^*) \\ &= 0. \end{aligned} \tag{5.10}$$

First, we consider the integration over $h_i^{\delta,\varepsilon}$:

$$\begin{aligned}\left|\int_{h_i^{\delta,\varepsilon}}(du^*-dv^*)\right| &\leqslant \int_{h_i^{\delta,\varepsilon}}(|d\mathbf{u}^*|+|d\mathbf{v}^*|)\\ &\leqslant \int_{h_i^{\delta,\varepsilon}} 2\,ds && \text{(by (5.3))}\\ &=2|h_i^{\delta,\varepsilon}|\\ &<2\delta && \text{(by (5.8)).} \qquad (5.11)\end{aligned}$$

We move on to consider

$$\int_{\gamma_i^{\delta,\varepsilon}}(du^*-dv^*).$$

Notice that u and v can be viewed as minimal graphs in $\mathbb{H}^2\times\mathbb{R}$ with $u|_{\gamma_i}=+\infty$ and $v|_{\gamma_i}=+\infty$. Combining this with Theorem 5.1, we see that, for any $\eta>0$, we may choose ε small enough so that

$$\left|\left\langle\frac{\nabla u}{\tau_u},\nu\right\rangle-1\right|<\eta \quad\text{and}\quad \left|\left\langle\frac{\nabla v}{\tau_v},\nu\right\rangle-1\right|<\eta \qquad (5.12)$$

on the lift $\gamma_i^{\delta,\varepsilon}$, where ν is the unit normal field of $\gamma_i^{\delta,\varepsilon}$ pointing toward γ_i. Hence,

$$\begin{aligned}&\left|\int_{\gamma_i^{\delta,\varepsilon}}(du^*-dv^*)\right|\\ &\quad=\left|\int_{\gamma_i^{\delta,\varepsilon}}\left(\left\langle\frac{\nabla u}{\tau_u},\nu\right\rangle-\left\langle\frac{\nabla v}{\tau_v},\nu\right\rangle\right)ds\right|\\ &\quad=\left|\int_{\gamma_i^{\delta,\varepsilon}}\left(\left(\left\langle\frac{\nabla u}{\tau_u},\nu\right\rangle-1\right)-\left(\left\langle\frac{\nabla v}{\tau_v},\nu\right\rangle-1\right)\right)ds\right|\\ &\quad\leqslant\int_{\gamma_i^{\delta,\varepsilon}}\left(\left|\left\langle\frac{\nabla u}{\tau_u},\nu\right\rangle-1\right|+\left|\left\langle\frac{\nabla v}{\tau_v},\nu\right\rangle-1\right|\right)ds\\ &\quad\leqslant\int_{\gamma_i^{\delta,\varepsilon}}2\eta && \text{(by (5.12))}\\ &\quad\leqslant 2\eta(|\gamma_i^{\delta}|+2\varepsilon). && (5.13)\end{aligned}$$

Combining (5.9)–(5.11) and (5.13), we get

$$\left|\int_{\omega^{\delta,\varepsilon}}\Psi\right|\leqslant\sum_{i=1}^{k}2M\delta+\sum_{i=1}^{k}2M\eta(|\gamma_i^{\delta}|+2\varepsilon).$$

Letting $\varepsilon\to0$, we have

$$\lim_{\varepsilon\to0}\left|\int_{\omega^{\delta,\varepsilon}}\Psi\right|\leqslant\sum_{i=1}^{k}2M\eta|\gamma_i^{\delta}|+\sum_{i=1}^{k}2M\delta.$$

The arbitrariness of η then implies that

$$\lim_{\varepsilon\to 0}\left|\int_{\omega^{\delta,\varepsilon}}\Psi\right|\leqslant 2kM\delta.$$

Therefore,

$$\left|\int_{\partial\Omega\cap U}\Psi\right|=\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\left|\int_{\omega^{\delta,\varepsilon}}\Psi\right|=0. \tag{5.14}$$

It remains to show that the distance function is constant. Suppose to the contrary that the distance function is not constant. Then, $M<\sup_{p\in\Omega}\operatorname{dist}(u(p),v(p))$. Let M' be a constant with $M<M'<\sup_{p\in\Omega}\operatorname{dist}(u(p),v(p))$. Consider a component Ω' of

$$\{p\in\Omega:\operatorname{dist}(u(p),v(p))>M'\}.$$

If Ω' admits a boundary component that encloses a disc in Ω, then, by case (i), we see that the distance function is a constant on Ω'. This contradicts the definition of Ω'. Hence, the distance function is constant. If Ω' admits a boundary component $\partial\Omega'\cap U$ which intersects $\bigcup_i\gamma_i$, then, by the discussion as for (5.14), we know that $\int_{\partial\Omega'\cap U}\Psi=0$. Consider the region A in Ω bounded by $\partial\Omega\cap U$ and $\partial\Omega'\cap U$. Approximating $\partial A\cap(\bigcup_i\gamma_i)$ similarly as above and applying the Stokes theorem to the approximating regions using (5.4), we see that the distance function on A is a constant. This contradicts the assumption that $\operatorname{dist}(u(p),v(p))=M$ for $p\in\partial\Omega\cap U$ while $\operatorname{dist}(u(p),v(p))=M'>M$ for $p\in\partial\Omega'\cap U$. Hence, the distance function is constant on Ω. (Note that the proof relies on the assumption that the distance function is identically M on $\partial\Omega\cap U$ and avoids the value M in Ω, but does not rely on the specific assumption that $\operatorname{dist}(u(p),v(p))>M$ for $p\in\Omega$. So the conclusion still holds if $\operatorname{dist}(u(p),v(p))<M$ for $p\in\Omega$ and $\operatorname{dist}(u(p),v(p))=M$ for $p\in\partial\Omega\cap U$. This observation, along with the analogous observation we made at the end of the discussion of case (i), will be used in case (iii-b).)

Case (iii): Ω is multiconnected.

Let ζ be a boundary component of Ω. Since U is a cylinder, there are two subcases: either ζ is homotopic to the core curve of U, or it encloses a simply connected domain in U.

Case (iii-a): ζ is homotopic to the core curve of U.

By Lemma 5.13, we see that u and v descend to two maps, say $u_U:U\to\mathbf{G}$ and $v_U:U\to\mathbf{G}$, from U to the graph $\mathbf{G}$. Notice that the cyclic order of the vertices $\{Q_i\}$ of $\mathbf{G}$ induces a cyclic order of all edges of $\mathbf{G}$ as follows:

$$E_1,\quad \overline{Q_1Q_2},\quad E_2,\quad \overline{Q_2Q_3},\quad \dots,\quad E_k,\quad \overline{Q_kQ_1},\quad E_1, \tag{5.15}$$

where E_i is the half-infinite edge attached to Q_i. If some edge $\overline{Q_iQ_{i+1}}$ has length zero, then we remove it from the above list, meaning that E_i and E_{i+1} are now consecutive. Consider the restrictions $u_U:\zeta\to\mathbf{G}$ and $v_U:\zeta\to\mathbf{G}$.

For any point p in the cusp neighbourhood W_i (defined in the proof of Lemma 5.11), we see that both $u(p)$ and $v(p)$ are contained in the union $E_i\cup\overline{Q_iQ_{i+1}}\cup E_{i+1}$: this union is isometric to $\mathbb{R}$. Therefore, $d_{\mathbf{G}}(u_U(p),v_U(p))=\operatorname{dist}(u(p),v(p))$ for any $p\in W_i$. Recall from the last paragraph of the proof of Lemma 5.11 that there is a component R_i of $U\setminus(\operatorname{Crit}(u)\cup\operatorname{Crit}(v))$ which contains γ_i in the boundary. In particular, for any point $p\in R_i$, we see that both $u(p)$ and $v(p)$ are contained in the half-infinite edge E_i. This implies that

$$d_{\mathbf{G}}(u_U(p),v_U(p))=\operatorname{dist}(u(p),v(p))\quad\text{for any } p\in R_i.$$

Hence,

$$d_{\mathbf{G}}(u_U(p),v_U(p))=\operatorname{dist}(u(p),v(p))\quad\text{for any } p\in U,$$

because $U=\bigcup_i(R_i\cup W_i)$. In particular, this equation of distances holds on ζ. For each i, let p_i be a point in $R_i\cap\zeta$. Combining the discussion above and the assumption that $\operatorname{dist}(u(\cdot),v(\cdot))$ is identically $M>0$ on ζ and the fact that

$$d_{\mathbf{G}}(u_U(\cdot),v_U(\cdot))=\operatorname{dist}(u(\cdot),v(\cdot))\quad\text{on } \zeta,$$

we see that

$$d_{\mathbf{G}}(u_U(p_i),Q_i)-d_{\mathbf{G}}(v_U(p_i),Q_i)$$

is well defined and has signs which are alternatively $+$ and $-$ with i.

In particular, this implies that $\mathbf{G}$ has even number of edges. Correspondingly, the crown end C in consideration also has an even number of ideal geodesic arcs.

It then follows from Lemma 5.12 that there are compatible choices of orientations of the restrictions $F_u|_U$ and $F_v|_U$, so that both $u-v$ and du^*-dv^* are well defined on U. We choose an orientation of $F_u|_U$ so that $u-v=M$ on ζ. Applying a similar argument as in case (ii), we see that $\int_\zeta(du^*-dv^*)=0$. Hence,

$$\int_\zeta\Psi=M\int_\zeta(du^*-dv^*)=0.$$

To prove that the distance function is a constant over Ω, let $M''>M$ be a constant which is close enough to M such that

$$\Omega'':=\{p\in\Omega:\operatorname{dist}(u(p),v(p))>M''\}$$

admits a boundary component, say $\partial\Omega''\cap\Omega$, which is homotopic to $\partial\Omega\cap U$. The discussion in the previous paragraphs yields that $\int_{\partial\Omega''\cap\Omega}\Psi=0$. Applying Stokes theorem to the cylinder region bounded by $\partial\Omega\cap U$ and $\partial\Omega''\cap\Omega$, and using (5.4) we see that the distance function is a constant on this cylinder region. This contradicts the assumption that the distance function is M on $\partial\Omega\cap U$ but is $M''>M$ on $\partial\Omega''\cap\Omega$. Hence, the distance function is constant over Ω.

Case (iii-b): ζ encloses a simply connected domain V on U.

In particular, the distance function is strictly less than M on V and identically M on $\partial V\cap U$. The arguments in cases (i) and (ii) still hold (see the notes at the end of the proof in those cases). Hence, similarly to cases (i) and (ii), we see that the distance function is constant on V. For any $r\geqslant M$, let Ω_r be the component of

$$\{p\in\Omega : \operatorname{dist}(u(p), v(p)) > r\}$$

that admits a complementary component, say V_r, which contains V. Let $\mathbf{r}$ be the supremum of $r\geqslant M$ such that V_r is simply connected. Then, for each $r<\mathbf{r}$, we see that the distance function is constant on V_r. By continuity, the distance function is constant on the closure $\overline{V}_{\mathbf{r}}$. On the other hand, the definition of $\mathbf{r}$ implies that for each $r>\mathbf{r}$, the complementary component V_r, as it contains the non-simply-connected $V_{\mathbf{r}}$ that is homotopic to the cylinder $U\supset\Omega$, is itself not simply connected. Then, the discussion in case (iii-a) implies that the distance function is constant on Ω_r, so is also constant on the union $\bigcup_{r>\mathbf{r}}\Omega_r$. Note that

$$\Omega = \bigcup_{r>\mathbf{r}} \Omega_r \cup (\overline{V}_{\mathbf{r}}\cap\Omega).$$

In summary, the distance function is constant on Ω. This completes the proof. □

5.5. Generalized maximum principle

We continue with the notation X_u, X_v, $f_u\colon X_u\to Y$, $f_v\colon X_v\to Y$, $\operatorname{Sing}(u)$, $\operatorname{Crit}(u)$, $\operatorname{Reg}(u)$, $\widetilde{\operatorname{Sing}}(u)$, $\widetilde{\operatorname{Crit}}(u)$, $\widetilde{\operatorname{Reg}}(u)$, $\operatorname{Sing}(v)$, $\operatorname{Crit}(v)$, $\operatorname{Reg}(v)$, $\widetilde{\operatorname{Sing}}(v)$, $\widetilde{\operatorname{Crit}}(v)$, and $\widetilde{\operatorname{Reg}}(v)$ introduced in the beginning of §5.3.

Recall that

- $\widetilde{\operatorname{Reg}}(u)$ is the subset of points $p\in\widetilde{Y}$ such that $u(p)$ is not a vertex of T;
- $\widetilde{\operatorname{Crit}}(u)$ is the subset of points $p\in\widetilde{Y}$ such that $u(p)$ is a vertex of T but admit a neighbourhood whose image under u is a geodesic segment;
- $\widetilde{\operatorname{Sing}}(u)$ is the subset of points $p\in\widetilde{Y}$ such that $u(p)$ is vertex and admit a neighbourhood whose image under u is a star with $m\geqslant 3$ edges/prongs.

LEMMA 5.15. *Let* $p \in Y$ *be a point.*

(a) *If* $p \in Y \setminus \mathrm{Sing}(u)$, *then there exists a neighbourhood* Ω *of* p *such that the component through* p *of the projection of* $u^{-1}(u(p))$ *into* Ω *cuts* Ω *into two sectors, each of which bounds an angle of* π *at* p.

(b) *If* $p \in \mathrm{Sing}(u)$ *is a singular point such that* $u(p)$ *is a vertex of valence* $m \geqslant 3$. *Then, there is a neighbourhood* $\Omega \subset Y$ *of* p *such that the component through* p *of the projection of* $u^{-1}(u(p))$ *into* Ω *cuts* Ω *into* m *sectors, each of which bounds an angle of* $2\pi/m$ *at* p.

Similar conclusions also hold for $\mathrm{Sing}(v)$.

Proof. Let $f_u: X_u \to Y$ be the harmonic map from the (closed) minimal graph to Y. Consider the horizontal foliation of the Hopf differential $\mathrm{Hopf}(f_u)$ near $f_u^{-1}(p)$.

For statement (a), there exists a neighbourhood $\Omega' \subset X_u$ of $f_u^{-1}(p)$ such that the component of $f_u^{-1} \circ u^{-1}(u(p)) \cap \Omega'$ containing $f_u^{-1}(p)$ is a smooth curve crossing Ω'. As f_u is a diffeomorphism, we see that $\Omega := f_u(\Omega')$ is cut out by the component of $u^{-1}(u(p)) \cap \Omega$ containing p into two sectors, each of which bounds an angle of π at p.

For statement (b), there exists a neighbourhood Ω' of $f_u^{-1}(p)$ cut out by the component of $f_u^{-1} \circ u^{-1}(u(p)) \cap \Omega'$ containing $f_u^{-1}(p)$ into m sectors, each of which bounds an angle of $2\pi/m$ at $f_u^{-1}(p)$ ([86, §6]). Since f_u is conformal at $f_u^{-1}(p)$ (because the Beltrami differential of f_u is zero at $f_u^{-1}(p)$), it follows that $\Omega := f_u(\Omega')$ is cut out by the component of $u^{-1}(u(p)) \cap \Omega$ containing p into m sectors, each of which bounds an angle of $2\pi/m$ at p. □

LEMMA 5.16. (Generalized maximum principle) *Let* $u: \widetilde{Y} \to T$ *and* $v: \widetilde{Y} \to T$ *be two minimal graphs. If there exists a point* $p \in \widetilde{Y}$ *which realizes the supremum of the distance function on* $\widetilde{Y}$, *i.e.*

$$\mathrm{dist}(u(p), v(p)) = \sup_{q \in \widetilde{Y}} \mathrm{dist}(u(q), v(q))$$

then $u = v$.

Remark 5.17. We will apply this lemma in the case when the two maps u and v share an admissible partial boundary map, but, as we do not need that hypothesis in the proof, we state the result in a more general form.

Proof. Let q be an arbitrary point on Y. Consider the images of a neighbourhood of q under u and v. There are two possibilities:

(a) for any neighbourhood Ω of q, the convex hull of $u(\Omega) \cup v(\Omega)$ is not a geodesic;

(b) q admits a neighbourhood Ω such that the convex hull of the union $u(\Omega) \cup v(\Omega)$ is a geodesic segment.

Notice that any singular point $q\in\mathrm{Sing}(u)\cup\mathrm{Sing}(v)$ satisfies the condition (a). Moreover, the subset of points satisfying the condition (b) is a proper open subset of Y, while the subset of points satisfying the condition (a) is a non-empty closed subset of Y.

Case (a): for any neighbourhood Ω of p, the convex hull of $u(\Omega)\cup v(\Omega)$ is not a geodesic.

Let Ω be a small neighbourhood of p such that both $u(\Omega)$ and $v(\Omega)$ are stable, meaning that smaller neighbourhoods share the same image as Ω, up to isotopies in T fixing $u(p)$ and $v(p)$, respectively. Suppose that $u(\Omega)$ (resp. $v(\Omega)$) is a star with $u(p)$ (resp. $v(p)$) as the vertex and $m_u\geqslant 2$ (resp. $m_v\geqslant 2$) edges. Let $\{\Omega_i(u):1\leqslant i\leqslant m_u\}$ be the set of complementary sectors in Ω of the connected locus of $u^{-1}(u(p))\cap\Omega$ containing p. Let $\{\Omega_j(v):1\leqslant j\leqslant m_v\}$ be defined similarly. There is a one-to-one correspondence between $\{\Omega_i(u)\}$ (resp. $\{\Omega_j(v)\}$) and the edges of $u(\Omega)$ (resp. $v(\Omega)$).

Suppose $u(p)\neq v(p)$ and $m_u=m_v=2$. There are two subcases depending on whether the convex hull of $u(\Omega)\cup v(\Omega)$ is a tripod or not. If it is not a tripod, then any point q in the non-empty set $(\Omega_1(u)\cup\Omega_2(u))\cap(\Omega_1(v)\cup\Omega_2(v))$ satisfies $d(u(q),v(q))>d(u(p),v(p))$, contradicting the assumption that p attains the supremum of $\mathrm{dist}(u(\cdot),v(\cdot))$ over Y. If the convex hull of $u(\Omega)\cup v(\Omega)$ is a tripod, then exactly one of $u(p)$ and $v(p)$ is a vertex of valence 3 of this tripod (since $v(p)\neq u(p)$). Without loss of generality, we may assume that $u(p)$ is the vertex. Then, there exists one of $\{\Omega_1(v),\Omega_2(v)\}$, say $\Omega_1(v)$, such that any point q in $\Omega_1(v)\cap(\Omega_1(u)\cup\Omega_2(u))$, which is non-empty by Lemma 5.15, satisfies $d(u(q),v(q))>d(u(p),v(p))$, contradicting the assumption that p attains the supremum of $\mathrm{dist}(u(\cdot),v(\cdot))$ over Y.

Suppose that $u(p)\neq v(p)$ and either m_u or m_v is bigger than 2. Without loss of generality, we may assume that $m_u>2$. Consider the stars $u(\Omega)$ and $v(\Omega)$. There exists at most one (open) edge of $u(\Omega)$, say $u(\Omega_1)$, such that any point of this edge to $v(\Omega)$ is strictly less than $d(u(p),v(p))$. Similarly, there exists at most one (open) edge of $v(\Omega)$, say $v(\Omega_1)$, such that the distance of any point of this edge to $u(\Omega)$ is strictly less than $d(u(p),v(p))$. In particular, if

$$\left(\bigcup_{i=2}^{m_u}\Omega_i(u)\right)\cap\left(\bigcup_{j=2}^{m_v}\Omega_j(v)\right)\neq\varnothing, \tag{5.16}$$

then any point in this set satisfies $d(u(q),v(q))>d(u(p),v(p))$, contradicting the assumption that p attains the supremum of $\mathrm{dist}(u(\cdot),v(\cdot))$ over Y. Next, we shall prove (5.16). By Lemma 5.15, the set $\Omega_1(u)$ bounds an angle of $2\pi/m_u\leqslant 2\pi/3$ while $\Omega_1(v)$ bounds an angle of $2\pi/m_v\leqslant\pi$. It follows that the closures (in Ω)

$$\overline{\bigcup_{i=2}^{m_u}\Omega_i(u)}\quad\text{and}\quad\overline{\bigcup_{i=2}^{m_v}\Omega_i(v)}$$

are sectors based at p of angles at least $\frac{4}{3}\pi$ and π, respectively. Consequently, the intersection set

$$\overline{\bigcup_{i=2}^{m_u} \Omega_i(u)} \cap \overline{\bigcup_{j=2}^{m_v} \Omega_j(v)}$$

has positive measure. On the other hand, the complement of

$$\bigcup_{i=2}^{m_u} \Omega_i(u) \quad \text{in} \quad \overline{\bigcup_{i=2}^{m_u} \Omega_i(u)}$$

is contained in $\mathrm{Crit}(u) \cap \Omega$, and hence has measure zero. Similarly, the complement of

$$\bigcup_{i=2}^{m_v} \Omega_i(v) \quad \text{in} \quad \overline{\bigcup_{i=2}^{m_v} \Omega_i(v)}$$

also has measure zero. Therefore,

$$\left(\bigcup_{i=2}^{m_v} \Omega_i(u)\right) \cap \left(\bigcup_{j=2}^{m_v} \Omega_j(v)\right) \neq \varnothing,$$

completing the proof of (5.16).

The discussion above excludes the possibility that $u(p) \neq v(p)$. Therefore, $u(p) = v(p)$. In particular, we have $d(u(p), v(p)) = 0$. The assumption that p attains the supremum of $\mathrm{dist}(u(\cdot), v(\cdot))$ over Y implies that $\mathrm{dist}(u(\cdot), v(\cdot)) \equiv 0$ on Y. Hence, $u = v$.

Case (b): p admits a neighbourhood Ω such that the convex hull of $u(\Omega) \cup v(\Omega)$ is a geodesic segment.

We notice that this geodesic admits two orientations; we pick one of these. This allows us to define two functions $\mathbf{u}$ and $\mathbf{v}$ on Ω as follows:

$$\mathbf{u}(q) := u(q) - u(p) \quad \text{and} \quad \mathbf{v}(q) := v(q) - v(p) \quad \text{for all } q \in \Omega.$$

In other words, both $\mathbf{u}$ and $\mathbf{v}$, restricted to Ω, may be viewed as minimal graphs in $\mathbb{H}^2 \times \mathbb{R}$ over a bounded simply connected domain. The maximum principle then implies that the distance function $\mathrm{dist}(u(\cdot), v(\cdot))$ is a constant on Ω. Then, again by the maximum principle, we see that the distance function $\mathrm{dist}(u(\cdot), v(\cdot))$ is a constant over the whole component, say Θ, of the subset of points satisfying condition (b) which contains p. By continuity, the distance function is also a constant over the closure $\overline{\Theta}$ of Θ. On the other hand, any point in the (interior) boundary $\partial\Theta \cap Y$, which is non-empty, satisfies condition (a). It then follows from Case (a) that $u = v$. This completes the proof. □

5.6. Finishing the proof of Theorem 5.7

Proof of Theorem 5.7. The theorem now follows from Lemmas 5.14 and 5.16. □

6. Subconvergence of harmonic maps rays

The goal of this section is to begin the proof of Theorem 1.1. Roughly that theorem asserts the convergence of some harmonic map rays to a Thurston geodesic. In this section, we prove the *sub*convergence of that family (see Theorem 1.3). We will be left to prove the (full) convergence, whose proof will occupy §7 and §8. Looking ahead, that proof of convergence will involve two steps: in §8, we will show that the harmonic map rays defined from our degenerating family X_t converge to a harmonic map ray from a punctured surface X_∞; This will be combined with the work of §7 that will show that such a harmonic map ray from X_∞, whose range is naturally a family of crowned surfaces, extends to a Thurston geodesic through closed surfaces.

6.1. Uniformly Lipschitz property of harmonic map rays

We begin by showing that the parametrization of a harmonic map ray captures a Lipschitz bound.

Lemma 6.1. (Uniformly Lipschitz) *Let $X\in\mathcal{T}(S)$ and $\Phi\in Q(X)$. Let*

$$\mathbf{HR}_{X,\Phi}(\cdot)\colon[0,\infty)\longrightarrow\mathcal{T}(S)$$

be the harmonic map ray (see Definition 3.1). Let $f_t\colon X\to\mathbf{HR}_{X,\Phi}(t)$ be the harmonic map at a fixed time t. Then, the map

$$f_s\circ f_t^{-1}\colon\mathbf{HR}_{X,\Phi}(t)\longrightarrow\mathbf{HR}_{X,\Phi}(s)$$

is $\sqrt{s/t}$-Lipschitz for all $s\geqslant t>0$.

Proof. By [96, Proposition 4.3], it follows that, for all $p\in M$ with $\Phi(p)\neq0$, $|\nu(p,t)|$ is an increasing function of $t\in(0,\infty)$, where $\nu(p,t)$ is the Beltrami differential of f_t.

Let

$$\mathcal{G}(p,t)=\log\left(\frac{1}{|\nu(p,t)|}\right).$$

Then, $\mathcal{G}$ is a decreasing function of $t\in(0,\infty)$ at p with $\Phi(p)\neq0$. Let $z=x+iy$ be a canonical coordinate chart of Φ near p. By (3.3), the pullback metric of $Y_t:=\mathbf{HR}_{X,\Phi}(t)$ by f_t on X is

$$f_t^*Y_t=2t(\cosh\mathcal{G}(z,t)+1)\,dx^2+2t(\cosh\mathcal{G}(z,t)-1)\,dy^2. \tag{6.1}$$

Then, for any $s>t>0$, since $\mathcal{G}$ is a decreasing function of $t\in(0,\infty)$, we have

$$\operatorname{Lip}(f_s\circ(f_t)^{-1})|_{f_t(p)}\leqslant\max\left\{\sqrt{\frac{2s(\cosh\mathcal{G}(p,s)+1)}{2t(\cosh\mathcal{G}(p,t)+1)}},\sqrt{\frac{2s(\cosh\mathcal{G}(p,s)-1)}{2t(\cosh\mathcal{G}(p,t)-1)}}\right\}<\sqrt{\frac{s}{t}}.$$

This implies that $f_s \circ (f_t)^{-1}$ is $\sqrt{s/t}$-Lipschitz outside the zero locus of Φ. By continuity, $f_s \circ (f_t)^{-1}$ is $\sqrt{s/t}$-Lipschitz on the whole surface Y_t. □

For $X, Y \in \mathcal{T}(S)$, let $\mathbf{HR}_{X,Y}: [1,+\infty) \to \mathcal{T}(S)$ be the harmonic map ray such that

$$\operatorname{Hopf}(X, \mathbf{HR}_{X,Y}(s)) = s \operatorname{Hopf}(X,Y).$$

In particular, $\mathbf{HR}_{X,Y}(1) = Y$. A direct consequence of Lemma 6.1 is the following.

Corollary 6.2. *Let Y be a fixed hyperbolic surface. For any $s>1$, the family of maps $\{\mathbf{HR}_{X,Y}: [1,s] \to \mathcal{T}(S)\}_{X \in \mathcal{T}(S)}$ is uniformly bounded and equi-continuous.*

6.2. Subconvergence of harmonic map rays

We first show that harmonic map rays are almost geodesics with respect to the Thurston metric. (We refer to the notion of length of a foliation defined in §2.3.)

Proposition 6.3. *For any $\varepsilon>0$, there exists $\mathbf{T}>0$ depending only on ε and the topology of S such that, for any harmonic map ray $\mathbf{HR}_{X,\Phi}$ with $\|\Phi\|=1$ and for all $s>t\geqslant \mathbf{T}$,*

$$\sqrt{\frac{s}{t}}(1-\varepsilon) \leqslant \frac{\ell_{Y_s}(\operatorname{Hor}(\Phi))}{\ell_{Y_t}(\operatorname{Hor}(\Phi))} \leqslant \sqrt{\frac{s}{t}} \tag{6.2}$$

and

$$\log\sqrt{\frac{s}{t}} - \varepsilon \leqslant d_{\mathrm{Th}}(Y_t, Y_s) \leqslant \log\sqrt{\frac{s}{t}}, \tag{6.3}$$

where

$$Y_t = \mathbf{HR}_{X,\Phi}(t) \quad \textit{and} \quad Y_s = \mathbf{HR}_{X,\Phi}(s).$$

Proof. Let $Y_s := \mathbf{HR}_{X,\Phi}(s)$ and let $\lambda := \operatorname{Hor}(\Phi)$ be the horizontal measured foliation of Φ. Then, the horizontal foliation of $s\Phi$ is $\sqrt{s}\lambda$. From (3.21) in Lemma 3.12, we see that

$$-C \leqslant \ell_{Y_s}(\sqrt{s}\lambda) - 2\|s\Phi\| \leqslant C$$

for some constant C depending only the topology of S. Combining with the assumption that $\|\Phi\|=1$, we infer that

$$-C \leqslant \ell_{Y_s}(\sqrt{s}\lambda) - 2s \leqslant C.$$

Then, for any $\varepsilon>0$, there exists $\mathbf{T}>0$ depending only on ε and the topology of S, such that for all $s \geqslant \mathbf{T}$, we have

$$e^{-\varepsilon} < \frac{\ell_{Y_s}(\sqrt{s}\lambda)}{2s} < e^{\varepsilon}.$$

In particular, for all $s>t\geqslant\mathbf{T}$,

$$\log\frac{\ell_{X_s}(\lambda)}{\ell_{X_t}(\lambda)}=\log\biggl(\frac{\sqrt{t}}{\sqrt{s}}\cdot\frac{\ell_{X_s}(\sqrt{s}\lambda)}{\ell_{X_t}(\sqrt{t}\lambda)}\biggr)\geqslant\log\biggl(\frac{\sqrt{t}}{\sqrt{s}}\cdot\frac{2s\cdot e^{-\varepsilon}}{2t\cdot e^{\varepsilon}}\biggr)\geqslant\log\sqrt{\frac{s}{t}}-2\varepsilon. \qquad (6.4)$$

Therefore,

$$d_{\mathrm{Th}}(X_t,X_s)=\log\sup_{\mu\in\mathcal{ML}(S)}\frac{\ell_{X_s}(\mu)}{\ell_{X_t}(\mu)}\geqslant\log\frac{\ell_{X_s}(\lambda)}{\ell_{X_t}(\lambda)}\geqslant\log\sqrt{\frac{s}{t}}-2\varepsilon.$$

On the other hand, by Lemma 6.1, we see that

$$d_{\mathrm{Th}}(Y_t,Y_s)\leqslant\log\sqrt{\frac{s}{t}}\quad\text{for all } s>t\geqslant1. \qquad \square$$

We now consider the compactness of a family of harmonic map rays passing through a fixed hyperbolic surface. The first part of the following theorem is a restatement of Theorem 1.3.

THEOREM 6.4. *For any fixed* $Y\in\mathcal{T}(S)$, *let* $X_n\in\mathcal{T}(S)$ *be any divergent sequence. Then, the sequence of harmonic map rays* $\mathbf{HR}_{X_n,Y}\colon[1,\infty)\to\mathcal{T}(S)$ *contains a subsequence which converges to some* (*reparameterized*) *Thurston geodesic locally uniformly.*

If, moreover, $\mathrm{Hor}(\mathrm{Hopf}(X_n,Y))$ *converges to some* $\lambda\in\mathcal{PML}(S)$ *as* $n\to\infty$, *then* λ *is a subset of the maximally stretched lamination of the limit geodesic.*

Proof. By Corollary 6.2, we see that $\mathbf{HR}_{X_n,Y}\colon[1,\infty)\to\mathcal{T}(S)$ contains a subsequence, still denoted by $\mathbf{HR}_{X_n,Y}$ for simplicity, which converges locally uniformly to some continuous map $\mathbf{R}\colon[1,\infty)\to\mathcal{T}(S)$. Let $\Phi_n:=\mathrm{Hopf}(X_n,Y)$ and $Y_{n,s}:=\mathbf{HR}_{X_n,Y}(s)$. Then, $\mathbf{R}(s)=\lim_{n\to\infty}Y_{n,s}$ and $\|\Phi_n\|\to\infty$, as $n\to\infty$.

Set $\hat{\Phi}_n:=\Phi_n/\|\Phi_n\|$ to be a quadratic differential on X_n of unit norm. Let

$$\mathbf{HR}_{X_n,\hat{\Phi}_n}\colon[\|\Phi_n\|,\infty)\longrightarrow\mathcal{T}(S)$$

be a reparametrization of the harmonic map ray $\mathbf{HR}_{X_n,Y}\colon[1,\infty)\to\infty$ by setting

$$\mathbf{HR}_{X_n,\hat{\Phi}_n}(s\|\Phi_n\|)=\mathbf{HR}_{X_n,Y}(s)$$

for $s\geqslant1$. In particular, $\mathbf{HR}_{X_n,\hat{\Phi}_n}(s\|\Phi_n\|)=Y_{n,s}\to\mathbf{R}(s)$ as $n\to\infty$. Since $\|\Phi_n\|\to\infty$, it follows that for any $\varepsilon>0$, there exists N_ε such that for all $n>N_\varepsilon$ we have $\|\Phi_n\|>\mathbf{T}$, where $\mathbf{T}$ is the constant from Proposition 6.3. Applying Proposition 6.3 to $\mathbf{HR}_{X_n,\hat{\Phi}_n}$, we see that, for all $n>N_\varepsilon$ and all $s>t>1$, we have

$$\log\sqrt{\frac{s}{t}}-\varepsilon\leqslant d_{\mathrm{Th}}(\mathbf{HR}_{X_n,\hat{\Phi}_n}(t\|\Phi_n\|),\mathbf{HR}_{X_n,\hat{\Phi}_n}(s\|\Phi_n\|))\leqslant\log\sqrt{\frac{s}{t}}.$$

By letting $n\to\infty$, we see that

$$\log\sqrt{\frac{s}{t}}-\varepsilon\leqslant d_{\mathrm{Th}}(\mathbf{R}(t),\mathbf{R}(s))\leqslant\log\sqrt{\frac{s}{t}}.$$

By the arbitrariness of ε, we see that

$$d_{\mathrm{Th}}(\mathbf{R}(t),\mathbf{R}(s))=\log\sqrt{\frac{s}{t}}.$$

This proves that $\mathbf{R}$ is a Thurston geodesic.

It remains to show that $\mathbf{R}$ maximally stretches λ. Let $\lambda_n:=\mathrm{Hor}(\Phi_n)$. Notice that $\|\Phi_n\|\to\infty$ as $n\to\infty$. It then follows from equation (6.2) that, for any $\varepsilon>0$, there exists N_ε such that, for all $n>N_\varepsilon$ and all $s>t>1$,

$$\sqrt{\frac{s}{t}}(1-\varepsilon)\leqslant\frac{\ell_{X_{n,s}}(\lambda_n)}{\ell_{X_{n,t}}(\lambda_n)}\leqslant\sqrt{\frac{s}{t}}.$$

Letting $n\to\infty$, we see that

$$\sqrt{\frac{s}{t}}(1-\varepsilon)\leqslant\frac{\ell_{X_s}(\lambda)}{\ell_{X_t}(\lambda)}\leqslant\sqrt{\frac{s}{t}},\tag{6.5}$$

where $X_s=\mathbf{R}(s)$ and $X_t=\mathbf{R}(t)$. The arbitrariness of ε then implies that

$$\frac{\ell_{X_s}(\lambda)}{\ell_{X_t}(\lambda)}=\sqrt{\frac{s}{t}}.\qquad\square$$

We end this subsection with a similar estimate for harmonic map dual rays.

PROPOSITION 6.5. *For any $\varepsilon>0$, there exists $\mathbf{T}>0$ depending only on ε and the topology of S such that, for any harmonic map dual ray $\mathbf{hr}_{Y,\lambda}$ with scaling $\ell_Y(\lambda)=1$ and for all $s>t\geqslant\mathbf{T}$,*

$$d_T(\mathbf{hr}_{Y,\lambda}(t),\mathbf{hr}_{Y,\lambda}(s))\geqslant\log\sqrt{\frac{s}{t}}-\varepsilon.\tag{6.6}$$

Proof. Let $X_t:=\mathbf{hr}_{Y,\lambda}(t)$ and $\Psi_t:=\mathrm{Hopf}(X_t,Y)$. Then, the horizontal measured foliation/lamination of Ψ_t is $t\lambda$. Hence, $\mathrm{Ext}_{X_t}(t\lambda)=\|\Psi_t\|$. Combining this with (3.21) in Lemma 3.12, we see that

$$-C\leqslant 2\,\mathrm{Ext}_{X_t}(t\lambda)-\ell_Y(t\lambda)\leqslant C$$

for some constant C depending only on the topology of S. Inserting the assumption that $\ell_Y(\lambda)=1$ into the above equation yields,

$$-C\leqslant 2\,\mathrm{Ext}_{X_t}(t\lambda)-t\leqslant C.$$

Then, for any $\varepsilon>0$, there exists $\mathbf{T}>0$ depending only on ε and the topology of S such that, for all $t\geqslant\mathbf{T}$, we have

$$e^{-\varepsilon}<\frac{t}{2\,\mathrm{Ext}_{X_t}(t\lambda)}<e^{\varepsilon}.$$

In particular, for all $s>t\geqslant\mathbf{T}$,

$$\begin{aligned} d_T(X_s,X_t) &= \frac{1}{2}\log\sup_{\mu\in\mathcal{ML}(S)}\frac{\mathrm{Ext}_{X_t}(\mu)}{\mathrm{Ext}_{X_s}(\mu)} \\ &\geqslant \frac{1}{2}\log\frac{\mathrm{Ext}_{X_t}(\lambda)}{\mathrm{Ext}_{X_s}(\lambda)} \\ &= \frac{1}{2}\log\left(\frac{s^2}{t^2}\frac{\mathrm{Ext}_{X_t}(t\lambda)}{\mathrm{Ext}_{X_s}(s\lambda)}\right) \\ &\geqslant \frac{1}{2}\log\left(\frac{s^2}{t^2}\frac{te^{-\varepsilon}/2}{se^{\varepsilon}/2}\right) \\ &\geqslant \log\sqrt{\frac{s}{t}}-\varepsilon. \end{aligned}$$

□

7. Construction of piecewise harmonic stretch maps

The goal of this section is to prove Theorem 1.5, which will be used in §8 to finish the proof of Theorem 1.1.

It is perhaps worth taking a moment to recall how this passage will sit in our general theory. In this section, we prove the existence of "piecewise harmonic stretch rays". These are paths that generalize Thurston's construction of concatenation of stretch lines when the maximally stretched lamination λ is not maximal. Here, instead of concatenating stretch maps with maximal laminations that extend λ, we use harmonic maps to define the stretch line on the complementary regions that are not ideal triangles.

On the other hand, we will later, in §12, construct a family of maps that extend the Thurston theory for these general laminations in a different way. In that section, given hyperbolic surfaces $Y,Z\in\mathcal{T}(S)$, a harmonic stretch ray will be a limit of a family of harmonic map rays from base points $X_n\in\mathcal{T}(S)$ that all proceed through Y to Z, as the base points X_n degenerate in $\mathcal{T}(S)$. In some sense, we will pick out, from some possible ways of stretching from Y to Z, a canonical path that minimizes a particular energy that we call the *harmonic stretch segment* from Y to Z.

With that context in mind, we now begin the discussion of the piecewise harmonic stretch maps.

7.1. Thurston's construction of stretch maps for maximal geodesic lamination

Given a closed hyperbolic surface Y and a maximal geodesic lamination λ, Thurston constructs stretch maps in two steps.

Step 1. The first step [90, Proposition 2.2] is to define a change in the hyperbolic structure of each ideal triangle complementary to λ on Y such that the sides of each triangle are expanded by a factor of e^t. The change is realized by a built-in Lipschitz map having Lipschitz constant e^t on the boundary and at most e^t in the interior. This is the only place where Thurston uses the assumption of λ being maximal. These changes match up along λ so that they change the arc length of the leaves of λ by a factor of e^t as well.

Step 2. The second step [90, §4] is to extend the (new) hyperbolic structures on the complement of λ over a neighbourhood $\mathcal{N}$ of λ by describing its developing map as an infinite product in the group of isometries of the hyperbolic plane, with the help of an induced measured foliation $F_{\mathcal{N}}(\lambda)$ on $\mathcal{N}$ transverse to λ. In this step, Thurston makes assumptions neither about the existence of transverse measures for λ, nor about the topological or combinatorial type of the complement of λ. In other words, this second step works for all geodesic laminations. The hyperbolic structure on $\mathcal{N}$ is determined by the transverse measured foliation $F_{\mathcal{N}}(\lambda)$ and a *sharpness function* in a neighbourhood of each ideal vertex (spike), which records the length of each horocycle leaf inside each spike of $Y\setminus\lambda$. The hyperbolic structure on $Y\setminus\lambda$ constructed in Step 1 provides a unique sharpness function. So the construction results in a unique hyperbolic structure on Y with a built-in Lipschitz homeomorphism. This step is carried out in detail in [72, §3] and [7, §5] (See also [11]).

To generalize the construction from the case of maximal laminations to non-maximal ones, the key point is to change the hyperbolic structure on each component of $Y\setminus\lambda$ in a suitable way. Here, we provide an approach to doing this by considering harmonic maps from punctured surfaces to crowned hyperbolic surfaces (possibly with simple closed geodesic boundary components).

7.2. Harmonic maps from punctured surfaces to crowned hyperbolic surfaces

The change of hyperbolic metric on $Y\setminus\lambda$ is based on the following theorem.

THEOREM 7.1. ([34, Theorem 1.2]) *Let X be a closed Riemann surface with a set of marked points $D=\{p_1, p_2, ..., p_k\}$ with fixed coordinate disks around them. For a collection of principal parts $\mathcal{P}=\{P_1, P_2, ..., P_k\}$ having poles of orders $n_i \geqslant 2$ for $i=1, 2, \dots k$,*

- *let $Q(X, D, \mathcal{P})$ be the space of meromorphic quadratic differentials on X with principal part P_i at p_i;*
- *let $\mathcal{T}(\mathcal{P})$ be the space of marked crowned hyperbolic surfaces homeomorphic to $X \setminus D$ with k crowns, each having $n_i - 2$ boundary cusps, with metric residues (see Definition 2.5) twice the absolute value of the real part of the residues of the principal parts P_i.*

Then, we have a homeomorphism

$$\Psi: \mathcal{T}(\mathcal{P}) \longrightarrow Q(X, D, \mathcal{P})$$

that assigns, to any marked crowned hyperbolic surface Y, the Hopf differential of the unique harmonic diffeomorphism in the appropriate homotopy class from $X \setminus D$ to Y with prescribed principal parts.

We will use Theorem 7.1 several times. For reader's convenience, we provide a sketch of the proof. Given a principal part P of order n on the punctured disc and a hyperbolic crown end with $n+2$ ideal geodesics whose metric residue is twice the absolute value of the real part of the residues of the principal parts P, Gupta [34, §3] constructed an asymptotic model harmonic map from the punctured disc to the idea polygon whose Hopf differential has P as the principal part at the puncture point $z=0$. Next, consider a punctured Riemann surface $X \setminus \{p_1, ..., p_k\}$ with a compact exhaustion $\{X_i\}$, a meromorphic differential with principal part $\{P_1, ..., P_k\}$ at p and a crowned hyperbolic surface Y with metric residue at the crown end corresponding to p_j twice the absolute value of the real part of the residues of the principal parts P_j. Then, using the constructed asymptotic model map to give boundary values on ∂X_i and solving the Dirichlet problem with these boundary values, we get a sequence of harmonic diffeomorphisms f_i from X_i to its image on Y. Generalizing the estimate developed in [98], Gupta proved that f_i converges to a unique harmonic diffeomorphism from $X \setminus \{p_1, ..., p_k\}$ to Y whose Hopf differential has principal part P_j near the puncture p_j. This defines the map $\Psi: \mathcal{T}(\mathcal{P}) \to Q(X, D, \mathcal{P})$ mentioned in Theorem 7.1. Since $\mathcal{T}(\mathcal{P})$ and $Q(X, D, \mathcal{P})$ are simply connected and have the same dimension, by Invariance of Domain, Gupta proved that Ψ is a homeomorphism by showing that Ψ is continuous, proper and injective.

Remarks 7.2. (1) The original first statement in [34, Theorem 1.2] is for $n_i \geqslant 3$, but the method also works for $n_i=2$, so we state the result more generally.

(2) Given an ideal hyperbolic polygon, there is more than one harmonic diffeomorphism from the complex plane to the prescribed hyperbolic polygon (see Remark 5.8).

The prescribed principal part at each puncture p_i in the statement of Theorem 7.1 is needed to ensure that the map Ψ is well defined.

(3) Let Φ be a meromorphic differential on $X\setminus D$ having poles of order at least 2 at each punctured/marked point. We take the principal parts of Φ as the prescribed principal part in Theorem 7.1. Then, by Theorem 7.1, there exists a unique (up to isometry homotopic to the identity) hyperbolic crowned surface Y, and a unique harmonic diffeomorphism $f: X\setminus D\to Y$ in the appropriate homotopy class whose Hopf differential is Φ.

7.3. Deformation of crowned hyperbolic surfaces

We start with the following lemma, an analogue of the Lemma 6.1.

LEMMA 7.3. (Uniformly Lipschitz) *Let $f_t: X\setminus D\to Y_t$ be a path of harmonic diffeomorphisms from a punctured Riemann surface $X\setminus D$ to crowned hyperbolic surfaces Y_t with Hopf differential $\Phi_t=t\Phi$. Then, for any $0<t<s$, the composition map*

$$f_s\circ(f_t)^{-1}: Y_t \longrightarrow Y_s$$

is a Lipschitz map having pointwise Lipschitz constant strictly less than $\sqrt{s/t}$. Moreover, it extends to an affine map of factor $\sqrt{s/t}$ from ∂Y_t to ∂Y_s.

We postpone the proof of Lemma 7.3 to the end of this section.

LEMMA 7.4. (Analyticity) *Let $f_t: X\setminus D\to Y_t$ be a path of harmonic diffeomorphisms from a punctured Riemann surface $X\setminus D$ to crowned hyperbolic surfaces Y_t with Hopf differential $\Phi_t=t\Phi$. Then, Y_t, f_t and the extended homeomorphism*

$$f_t\circ(f_1)^{-1}: \overline{Y}_1 \longrightarrow \overline{Y}_t$$

(*obtained by Lemma* 7.3) *are real analytic in $t>0$.*

We postpone the proof of Lemma 7.4 to the end of §8.1. (It is important to note here that we will use the other statements of Theorem 1.5 in the arguments in the earlier part of §8.1.)

With the notation as in Lemma 7.4, let l be a proper and bi-infinite vertical leaf of Φ. The image $f_t(l)$ is a proper and bi-infinite arc on Y_t. For each fixed $t>0$, by Theorem 3.9, as the distance from zeros of $t\Phi$ goes to infinity, the images of horizontal leaf segments of $t\Phi$ converge smoothly to segments on the boundary ∂Y_t. Hence, both ends of $f_t(l)$ approach ∂Y_t perpendicularly.

7.4. Extending foliations across a geodesic lamination

Lemma 7.3 allows us to deform crowned hyperbolic surfaces in a controlled way. To deform a closed hyperbolic surface, we need to "glue" those deformed crowned hyperbolic surfaces to get a closed hyperbolic surface. This gluing process relies on extending a foliation across a geodesic lamination (Lemma 7.7). The following lemma is key to that extension.

LEMMA 7.5. (Lipschitz line field) *Let $f: X\setminus D\to Y$ be a harmonic diffeomorphism from a punctured Riemann surface $X\setminus D$ to a crowned hyperbolic surface Y with Hopf differential Φ. Then, there exist constants $R_0>0$ and $L=L(R_0)$ such that, for all $R>R_0$, the line field tangent to the pushforward of the vertical leaves of Φ on $Y\setminus f(\mathscr{P}_R)$ is L-Lipschitz, where $\mathscr{P}_R$ is Minsky's polygonal region of Φ (see Theorem 3.7).*

Proof. Let $\mathbf{h}$ and $\mathbf{v}$ be respectively the line field tangent to the horizontal and vertical foliations of Φ on $X\setminus\mathrm{Zero}(\Phi)$, where $\mathrm{Zero}(\Phi)$ represents the zero set of Φ. Let $p_0\in X\setminus\mathscr{P}_R$. Let $\widetilde{X}$ and $\widetilde{Y}$ be the universal cover of X and Y respectively. Let $\widetilde{\mathbf{h}}$, $\widetilde{\mathbf{v}}$, $\tilde{\Phi}$, $\tilde{f}$ and $\tilde{p}_0$ be respectively lifts of $\mathbf{h}$, $\mathbf{v}$, Φ, f, and p_0. Choosing orientation representatives for $\widetilde{\mathbf{h}}$ and $\widetilde{\mathbf{v}}$, we may assume that $\widetilde{\mathbf{h}}$ and $\widetilde{\mathbf{v}}$ are unit vector fields with respect to the flat $|\tilde{\Phi}|$-metric. To simplify notations, we also denote by $\widetilde{\mathbf{h}}$ and $\widetilde{\mathbf{v}}$ their pushforward to $\widetilde{Y}$ by $\tilde{f}$. Let $|\widetilde{\mathbf{h}}|_{\widetilde{Y}}$ and $|\widetilde{\mathbf{v}}|_{\widetilde{Y}}$ be their pointwise length with respect to the hyperbolic metric $\widetilde{Y}$. Finally, let $\widetilde{\nabla}^{\widetilde{Y}}$ be the covariant derivative with respect to $\widetilde{Y}$. To prove the lemma, it suffices to prove that the gradient $\widetilde{\nabla}^{\widetilde{Y}}(\widetilde{\mathbf{v}}/|\widetilde{\mathbf{v}}|_{\widetilde{Y}})$ is bounded on the lift of $X\setminus\mathscr{P}_R$. Since $\widetilde{\mathbf{h}}/|\widetilde{\mathbf{h}}|_{\widetilde{Y}}$ and $\widetilde{\mathbf{v}}/|\widetilde{\mathbf{v}}|_{\widetilde{Y}}$ are orthogonal to each other (and hence define a frame at each point $X\setminus\mathrm{Zero}(\Phi)$), it is equivalent to prove the following two properties:

(i) $\big\|\widetilde{\nabla}^{\widetilde{Y}}_{\widetilde{\mathbf{v}}/|\widetilde{\mathbf{v}}|_{\widetilde{Y}}}(\widetilde{\mathbf{v}}/|\widetilde{\mathbf{v}}|_{\widetilde{Y}})\big\|_{\widetilde{Y}}$ is bounded from above;

(ii) $\big\|\widetilde{\nabla}^{\widetilde{Y}}_{\widetilde{\mathbf{h}}/|\widetilde{\mathbf{h}}|_{\widetilde{Y}}}\widetilde{\mathbf{v}}/|\widetilde{\mathbf{v}}|_{\widetilde{Y}})\big\|_{\widetilde{Y}}$ is bounded from above.

Notice that $\widetilde{\nabla}^{\widetilde{Y}}_{\widetilde{\mathbf{v}}/|\widetilde{\mathbf{v}}|_{\widetilde{Y}}}(\widetilde{\mathbf{v}}/|\widetilde{\mathbf{v}}|_{\widetilde{Y}})$ is the geodesic curvature of vertical leaves with respect to the hyperbolic metric on $\widetilde{Y}$. Choose the local coordinate near $\tilde{p}_0$ such that $\tilde{\Phi}=dz^2$. By (3.3), the pullback of the hyperbolic metric on $\widetilde{Y}$ near $\tilde{p}_0$ is

$$2(\cosh\mathscr{G}(z)+1)\,dx^2+2(\cosh\mathscr{G}(z)-1)\,dy^2,$$

where $\mathscr{G}(p)=\log(1/|\nu(p)|)$ and $\nu(p)$ is the Beltrami differential of $\tilde{f}$. Let

$$g_{11}=2(\cosh\mathscr{G}(z)+1),\quad g_{12}=g_{21}=0,\quad g_{22}=2(\cosh\mathscr{G}(z)-1)$$

and

$$g^{11}=\frac{1}{2(\cosh\mathscr{G}(z)+1)},\quad g^{12}=g^{21}=0,\quad g^{22}=\frac{1}{2(\cosh\mathscr{G}(z)-1)}.$$

Accordingly, the Christoffel symbols of the second type

$$\Gamma_{ij}^{m}=\frac{1}{2}\sum_{l=1}^{2}g^{lm}(\partial_j g_{il}+\partial_i g_{jl}-\partial_l g_{ji})$$

are

$$\begin{aligned}
&\Gamma_{11}^{1}=\frac{1}{2}g^{11}\partial_1 g_{11}=\frac{\sinh\mathscr{G}}{2(\cosh\mathscr{G}+1)}\frac{\partial\mathscr{G}}{\partial x}, &&\Gamma_{11}^{2}=-\frac{1}{2}g^{22}\partial_2 g_{11}=-\frac{\sinh\mathscr{G}}{2(\cosh\mathscr{G}-1)}\frac{\partial\mathscr{G}}{\partial y},\\
&\Gamma_{12}^{1}=\frac{1}{2}g^{11}\partial_2 g_{11}=\frac{\sinh\mathscr{G}}{2(\cosh\mathscr{G}+1)}\frac{\partial\mathscr{G}}{\partial y}, &&\Gamma_{12}^{2}=\frac{1}{2}g^{22}\partial_1 g_{22}=\frac{\sinh\mathscr{G}}{2(\cosh\mathscr{G}-1)}\frac{\partial\mathscr{G}}{\partial x},\\
&\Gamma_{21}^{1}=\Gamma_{12}^{1}, &&\Gamma_{21}^{2}=\Gamma_{12}^{2}\\
&\Gamma_{22}^{1}=-\frac{1}{2}g^{11}\partial_1 g_{22}=-\frac{\sinh\mathscr{G}}{2(\cosh\mathscr{G}+1)}\frac{\partial\mathscr{G}}{\partial x}, &&\Gamma_{22}^{2}=\frac{1}{2}g^{22}\partial_2 g_{22}=\frac{\sinh\mathscr{G}}{2(\cosh\mathscr{G}-1)}\frac{\partial\mathscr{G}}{\partial y}.
\end{aligned}$$

Note that, with respect to the chosen coordinates, the vector fields are

$$\frac{\tilde{\mathbf{h}}}{|\tilde{\mathbf{h}}|_{\widetilde{Y}}}=\frac{1}{\sqrt{g_{11}}}\partial_1 \quad\text{and}\quad \frac{\tilde{\mathbf{v}}}{|\tilde{\mathbf{v}}|_{\widetilde{Y}}}=\frac{1}{\sqrt{g_{22}}}\partial_2, \tag{7.1}$$

where $\partial_1:=\partial/\partial_x$ and $\partial_2:=\partial/\partial_y$ are coordinate vector fields. Hence,

$$\begin{aligned}
\widetilde{\nabla}^{\widetilde{Y}}_{\tilde{\mathbf{v}}/|\tilde{\mathbf{v}}|_{\widetilde{Y}}}\left(\frac{\tilde{\mathbf{v}}}{|\tilde{\mathbf{v}}|_{\widetilde{Y}}}\right)&=\widetilde{\nabla}^{\widetilde{Y}}_{\frac{1}{\sqrt{g_{22}}}\partial_2}\left(\frac{1}{\sqrt{g_{22}}}\partial_2\right)\\
&=\frac{1}{g_{22}}\Gamma_{22}^{1}\partial_1+\frac{1}{\sqrt{g_{22}}}\left(\partial_2\frac{1}{\sqrt{g_{22}}}+\frac{1}{\sqrt{g_{22}}}\Gamma_{22}^{2}\right)\partial_2\\
&=\frac{1}{g_{22}}\Gamma_{22}^{1}\partial_1.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left\|\widetilde{\nabla}^{\widetilde{Y}}_{\tilde{\mathbf{v}}/|\tilde{\mathbf{v}}|_{\widetilde{Y}}}\left(\frac{\tilde{\mathbf{v}}}{|\tilde{\mathbf{v}}|_{\widetilde{Y}}}\right)\right\|_{\widetilde{Y}}&=\left\|\frac{1}{g_{22}}\Gamma_{22}^{1}\partial_1\right\|_{\widetilde{Y}}\\
&=\frac{\sqrt{g_{11}}}{g_{22}}|\Gamma_{22}^{1}|\\
&=\frac{\sqrt{2(\cosh\mathscr{G}+1)}}{2(\cosh\mathscr{G}-1)}\frac{\sinh\mathscr{G}}{2(\cosh\mathscr{G}+1)}\left|\frac{\partial\mathscr{G}}{\partial x}\right|\\
&=\frac{1}{\sqrt{2(\cosh\mathscr{G}+1)}}\frac{\sinh\mathscr{G}}{2(\cosh\mathscr{G}-1)}\left|\frac{\partial\mathscr{G}}{\partial x}\right| \qquad (7.2)\\
&\leqslant\frac{1}{4\sinh(\mathscr{G}/2)}|\nabla\mathscr{G}|, \qquad (7.3)
\end{aligned}$$

where $\nabla\mathscr{G}$ represents the gradient of $\mathscr{G}$ with respect to the flat $|\Phi|$-metric. Similarly,

$$\begin{aligned}\widetilde{\nabla}^{\widetilde{Y}}_{\tilde{\mathbf{h}}/|\tilde{\mathbf{h}}|_{\widetilde{Y}}}\left(\frac{\widetilde{\mathbf{v}}}{|\widetilde{\mathbf{v}}|_{\widetilde{Y}}}\right) &= \widetilde{\nabla}^{\widetilde{Y}}_{\frac{1}{\sqrt{g_{11}}}\partial_1}\left(\frac{1}{\sqrt{g_{22}}}\partial_2\right)\\ &= \frac{1}{\sqrt{g_{11}g_{22}}}\Gamma^1_{12}\partial_1+\frac{1}{\sqrt{g_{11}}}\left(\partial_1\frac{1}{\sqrt{g_{22}}}+\frac{1}{\sqrt{g_{22}}}\Gamma^2_{12}\right)\partial_2\\ &= \frac{1}{\sqrt{g_{11}g_{22}}}\Gamma^1_{12}\partial_1\end{aligned}$$

and

$$\begin{aligned}\left\|\widetilde{\nabla}^{\widetilde{Y}}_{\tilde{\mathbf{h}}/|\tilde{\mathbf{h}}|_{\widetilde{Y}}}\left(\frac{\widetilde{\mathbf{v}}}{|\widetilde{\mathbf{v}}|_{\widetilde{Y}}}\right)\right\|_{\widetilde{Y}} &= \left\|\frac{1}{\sqrt{g_{11}g_{22}}}\Gamma^1_{12}\partial_1\right\|_{\widetilde{Y}}\\ &= \frac{1}{\sqrt{g_{22}}}|\Gamma^1_{12}|\\ &= \frac{1}{\sqrt{2(\cosh\mathscr{G}-1)}}\frac{\sinh\mathscr{G}}{2(\cosh\mathscr{G}+1)}\left|\frac{\partial\mathscr{G}}{\partial y}\right| \qquad (7.4)\\ &\leqslant \frac{1}{4\cosh(\mathscr{G}/2)}|\nabla\mathscr{G}|. \qquad (7.5)\end{aligned}$$

In the remainder of the proof, we shall estimate $|\nabla\mathscr{G}|$ and $|\nabla\mathscr{G}|/\mathscr{G}$. By Lemma 3.5, we see that

$$\mathscr{G}(\tilde{p}_0)\leqslant\frac{\sinh^{-1}(|\chi(X)|/(R-4)^2)}{\exp(R-4)} \qquad (7.6)$$

holds for every $\tilde{p}_0\in\widetilde{X\setminus\mathscr{P}_R}$. Together with (7.3) and (7.5), this implies that there exists $R_0>0$ depending only on the topology of X such that

$$\left\|\widetilde{\nabla}^{\widetilde{Y}}_{\tilde{\mathbf{v}}/|\tilde{\mathbf{v}}|_{\widetilde{Y}}}\left(\frac{\widetilde{\mathbf{v}}}{|\widetilde{\mathbf{v}}|_{\widetilde{Y}}}\right)\right\|_{\widetilde{Y}}\leqslant\frac{|\nabla\mathscr{G}|}{\mathscr{G}} \quad\text{and}\quad \left\|\widetilde{\nabla}^{\widetilde{Y}}_{\tilde{\mathbf{h}}/|\tilde{\mathbf{h}}|_{\widetilde{Y}}}\left(\frac{\widetilde{\mathbf{v}}}{|\widetilde{\mathbf{v}}|_{\widetilde{Y}}}\right)\right\|_{\widetilde{Y}}\leqslant|\nabla\mathscr{G}| \qquad (7.7)$$

hold for every $\tilde{p}_0\in\widetilde{X\setminus\mathscr{P}_{R_0}}$. Combining Lemma 3.6 and (7.6), we see that there exists positive constants $c_1=c_1(R_0)$ and $c_2=c_2(R_0)$ depending only on R_0 and the topology of X such that

$$|\nabla\mathscr{G}(\tilde{p}_0)|\leqslant c_1 \quad\text{and}\quad \frac{|\nabla\mathscr{G}|}{\mathscr{G}}(\tilde{p}_0)\leqslant c_2 \qquad (7.8)$$

for every $\tilde{p}_0\in\widetilde{X\setminus\mathscr{P}_{R_0}}$. It then follows from (7.7) and (7.8) that, for any $R>R_0$, using that $\mathscr{P}_{R_0}\subset\mathscr{P}_R$, the line field $\widetilde{\mathbf{v}}/|\widetilde{\mathbf{v}}|_{\widetilde{Y}}$ is $\max\{c_1,c_2\}$-Lipschitz on the lift of $f(X\setminus\mathscr{P}_R)$ to the universal cover $\widetilde{Y}$. Hence, the line field $\mathbf{v}/|\mathbf{v}|_Y$ is $\max\{c_1,c_2\}$-Lipschitz on

$$f(X\setminus\mathscr{P}_R)=Y\setminus f(\mathscr{P}_R).$$

This completes the proof. □

Remark 7.6. Theorem 3.9 also holds in the current setting of punctured Riemann surfaces and crowned hyperbolic surfaces.

- By [97, Lemma 2.2 (v)], the geodesic curvature of the pushforward of the horizontal leaves of Φ in $Y\setminus f(\mathscr{P}_R)$ goes to zero uniformly as $R\to\infty$. Using the above computation, the horizontal geodesic curvature is

$$\left\|\widetilde{\nabla}^{\widetilde{Y}}_{\widetilde{\mathbf{h}}/|\widetilde{\mathbf{h}}|_{\widetilde{Y}}}\left(\frac{\widetilde{\mathbf{h}}}{|\widetilde{\mathbf{h}}|_{\widetilde{Y}}}\right)\right\|_{\widetilde{Y}} = \frac{\sqrt{g_{22}}}{g_{11}}|\Gamma^2_{11}| = \frac{\sinh\mathscr{G}}{2(\cosh\mathscr{G}+1)\sqrt{2(\cosh\mathscr{G}-1)}}\left|\frac{\partial\mathscr{G}}{\partial y}\right| \leqslant \frac{1}{4\cosh(\mathscr{G}/2)}\left|\nabla\mathscr{G}\right|$$

which goes to zero uniformly as $R\to\infty$, by (7.6) and (7.8).

- By (7.6) and (3.3), the hyperbolic length of every vertical leaf of Φ on $Y\setminus f(\mathscr{P}_R)$ converges to zero exponentially as $R\to\infty$. In particular, for any $\varepsilon>0$, there is a constant R such that $Y\setminus f(\mathscr{P}_R)$ is contained in the ε-neighborhood of the boundary of Y.

LEMMA 7.7. (Gluing partial foliations) *Let $Y\in\mathcal{T}(S)$ be any closed hyperbolic surface, and let λ be any geodesic lamination. Then, for any harmonic diffeomorphism $f: X\to Y\setminus\lambda$ from some (possibly disconnected) punctured surface X, the pushforward of the vertical measured foliation of the Hopf differential of f on $Y\setminus\lambda$ extends to a unique measured foliation on Y.*

Proof. Let Φ be the Hopf differential of f. Let V be the pushforward of the vertical measured foliation of Φ on $Y\setminus\lambda$. Let $\mathbf{v}$ be the line field tangent to $V_{Y\setminus\lambda}$, outside the images of zeros of Φ under f, with respect to the hyperbolic metric on Y. By Lemma 7.5, we know that there exist constants $R>0$ and $c>0$ such that $\mathbf{v}$ is c-Lipschitz on

$$f(X\setminus\mathscr{P}_R)=Y\setminus(\lambda\cup f(\mathscr{P}_R)),$$

where $\mathscr{P}_R$ is Minsky's polygonal region of Φ.

Let H be the pushforward of the horizontal measured foliation of Φ on $Y\setminus\lambda$. By [97, Lemma 2.2 (v)] (see also Remark 7.6), as R goes to infinity, the hyperbolic geodesic curvature of leaves of H on

$$f(X\setminus\mathscr{P}_{R_0})=Y\setminus(\lambda\cup f(\mathscr{P}_R))$$

converges to zero uniformly. Combining this with Theorem 3.9 (see also the second item in Remark 7.6) and the assumption $f(X)=Y\setminus\lambda$, we see that the line field tangent to leaves of H on $Y\setminus(\lambda\cup f(\mathscr{P}_R))$ extends continuously to $Y\setminus f(\mathscr{P}_R)$ by assigning to each point p in λ the line of tangent direction to λ at p. On the other hand, by (3.2), (the pushforward of) leaves of H and V are orthogonal to each other. Hence, the restriction to $Y\setminus(\lambda\cup f(\mathscr{P}_R))$ of the line field $\mathbf{v}$ continuously extends to a line field $\mathbf{v}'$ on $Y\setminus f(\mathscr{P}_R)$ by assigning to each point p in λ the line of normal direction to λ. Notice that the restriction to λ of the extended line field $\mathbf{v}'$ is Lipschitz (with respect to the hyperbolic metric on Y instead of the intrinsic metric on λ). By Lemma 7.5, the restriction to $Y\setminus(\lambda\cup f(\mathscr{P}_R))$ of the extended line field $\mathbf{v}'$ is also Lipschitz. Hence, the extended line field $\mathbf{v}'$ is Lipschitz on $Y\setminus f(\mathscr{P}_R)$. Therefore, we can continuously extend the partial foliation $V_{Y\setminus(\lambda\cup\mathscr{P}_R)}$ to a unique foliation $V_{Y\setminus\mathscr{P}_R}$ on $Y\setminus\mathscr{P}_R$ by integrating the extended line field $\mathbf{v}'$. Together with the partial foliation $V_{f(\mathscr{P}_R)}$, this extends the foliation V, originally defined on $Y\setminus\lambda$, to a unique foliation V' on the whole surface Y. Recall that V admits a transverse measure arising from the vertical transverse measure of Φ. Define a transverse measure for V' near λ as follows: for any arc η near λ and transverse to λ, we project η to a segment $\hat{k}$ of λ along leaves of V' and define intersection number between η and V' to be half the hyperbolic length of the geodesic segment $\hat{k}$. By Theorem 3.9, this construction is well defined. Again by Theorem 3.9, these two transverse measures coincide on $Y\setminus(\lambda\cup f(\mathscr{P}_R))$. Hence, we construct a transverse measure on V' whose restriction to $Y\setminus\lambda$ is exactly the transverse measure of V. This completes the proof. □

For convenience of a later reference, we summarize some basic properties of the extended vertical foliation as follows, where the first conclusion is proved in the second paragraph of the proof of Lemma 7.7 and the second conclusion follows from Theorem 3.9 (also follows from (3.3), Lemma 3.5 and Remark 7.6).

LEMMA 7.8. *Let V' be the extended measured foliation in Lemma 7.7. Then, V' is transverse to λ and intersects λ orthogonally at every intersection point. Furthermore, for any arc κ in a small neighborhood of λ that is transverse to V', the intersection number $i(\kappa, V')$ equals half of the hyperbolic length of projection of κ to a leaf of λ along leaves of V'.*

7.5. Proof of Theorem 1.5

We now construct piecewise harmonic stretch lines based on Lemmas 7.3, 7.5 and 7.7, proving Theorem 1.5.

Proof of Theorem 1.5. The proof is similar to the proof of [90, Corollary 4.2], except that we use Lemma 7.3 to deform the hyperbolic metrics on each component of $S\setminus\lambda$. Let $Y\in\mathcal{T}(S)$ be a closed hyperbolic surface and λ a geodesic lamination on Y. Let $Y^1, ..., Y^m$ be the connected components of $Y\setminus\lambda$, which are crowned hyperbolic surfaces. Suppose that Y^i has k_i crowns. For each $1\leqslant i\leqslant m$, let us choose a closed Riemann surface X^i

with marked points $D^i=\{p_1^i,...,p_{k_i}^i\}$ and a harmonic diffeomorphism $f^i: X^i \to Y^i$ with the Hopf differential denoted by Φ^i. Note that Φ^i has a pole of order $r_j^i \geqslant 2$ at each marked point p_j^i of X^i. For each $1\leqslant i\leqslant m$ and each $t>0$, let $\mathcal{T}_i(t)$ be the family of spaces of crowned hyperbolic surfaces homeomorphic to X^i with k_i crowns, each having r_j^i-2 boundary cusps, with metric residue being twice of the absolute value of the real part of the residues of the principal parts of $t\Phi^i$. Applying Theorem 7.1 to the pair $(X^i, t\Phi^i)$ and taking the principal part of $t\Phi^i$ at each puncture of X^i as the prescribed principal part of Theorem 7.1, we see that, for each $t>0$, there exist a unique crowned hyperbolic surface Y_t^i in $\mathcal{T}_i(t)$ and a unique harmonic diffeomorphism $f_t^i: X^i \to Y_t^i$ with $Y_1^i=Y^i$ such that $\mathrm{Hopf}(f_t^i)=t\,\mathrm{Hopf}(f^i)=t\Phi^i$. This proves item (a) of Theorem 1.5.

Next, we shall glue $Y_t^1,...,Y_t^m$ to obtain a family of closed hyperbolic surfaces $\{Y_t\}$ as follows. For $t>0$, let V_t^i be the pushforward on Y_t^i of the vertical foliation of the Hopf differential of Φ^i via $f_t^i: X^i\to Y_t^i$. For $t=1$, by Lemma 7.7, there exists a unique measured foliation V_1 on Y such that $V_1|_{Y^i}=V_1^i$. For $t>0$, by Lemma 7.3, the map

$$u_{1,t}^i := f_t^i \circ (f_1^i)^{-1}: Y_1^i \longrightarrow Y_t^i$$

extends to an affine map of linear factor $\sqrt{t}$ on the boundary. Notice that there are many hyperbolic surfaces $Y_t'\in\mathcal{T}(S)$ whose restriction to the component corresponding Y_1^i is Y_t^i. Any two such metrics differ by a "shearing" along the geodesic lamination λ. Constructing the desired hyperbolic metric is equivalent to specifying the "shearing" along λ. This is done through the measured foliation V_1 as follows. From the construction of V_t^i, we see that $(u_{1,t}^i)^*V_t^i=\sqrt{t}V_1^i$. This allows us to glue Y_t^i in such a way that the extended foliation on the resulting surface arising from $V_t^i,...,V_t^m$ by Lemma 7.7 is exactly $\sqrt{t}V_1$. Let Y_t be the resulting hyperbolic surface and V_t be the extended measured foliation arising from $V_t^1,...,V_t^m$ by Lemma 7.7. (Later in this proof we will compute the monodromy of Y_t in terms of V_t^i and $V_t=\sqrt{t}V_1$. In particular, this demonstrates the uniqueness of Y_t.)

Having defined the surface Y_t, we next wish to extend the maps $u_{1,t}^i$ to a map $u_{1,t}: Y_1\to Y_t$. To do this note that, for any sufficiently small neighborhood U of λ on Y_t, two leaf segments of the restriction to U of $V_t^1,...,V_t^m$ are contained in a single leaf of the restriction to U of V_t if and only if their preimages under $u_{1,t}^1,...,u_{1,t}^m$ are contained in a single leaf of the restriction to $\lambda\cup(u_{1,t}^1)^{-1}(U\setminus\lambda)\cup...\cup(u_{1,t}^m)^{-1}(U\setminus\lambda)$ of V_1. Therefore, the maps $u_{1,t}^1,...,u_{1,t}^m$ may be glued together to get a homeomorphism $u_{1,t}$ from Y_1 to Y_t that sends leaves of V_1 to V_t; note that then $(u_{1,t})^*V_t=\sqrt{t}V_1$.

By Lemma 7.3, we see that for any $0<s<t$, the composition map $u_{s,t}^i := f_t^i\circ(f_s^i)^{-1}$: $Y_s^i\to Y_t^i$ is a Lipschitz map having pointwise Lipschitz constant strictly less than $\sqrt{t/s}$ and extends to an affine map of factor $\sqrt{t/s}$ from ∂Y_s^i to ∂Y_t^i. Note that

$$u_{s,t}^i = u_{1,t}^i \circ (u_{1,s}^i)^{-1}.$$

Combining with the discussion in the preceding paragraph, we see that we may extend $u^1_{s,t}, ..., u^m_{s,t}$ to obtain a homeomorphism $u_{s,t} := u_{1,t} \circ (u_{1,s})^{-1}: Y_s \to Y_t$ which is an affine map of factor $\sqrt{t/s}$ on λ and which has pointwise Lipschitz constant strictly less than $\sqrt{t/s}$ in $Y \setminus \lambda$. This proves item (b) of Theorem 1.5.

We now compute the monodromy of Y_t and prove the real analytic dependence on t (assuming Lemma 7.4), following the idea of Thurston [90, §4] (see also [6, Theorem A] or [72, §3.5 in Chapter 2]). We start with an overview of the argument. Notice that the length functions of simple closed geodesics are real analytic over $\mathcal{T}(S)$ and that each hyperbolic surface is uniquely determined by the lengths of finitely many simple closed geodesics ([77], [67]), hence by the monodromy matrices of finitely many simple closed curves. In view of this, to show that Y_t varies real analytically on t, it suffices to show that the monodromy matrix of any simple closed curve varies real analytically in t. Let α be an arbitrary simple closed curve. In [90], Thurston realizes α as a concatenation of geodesic subarcs of (isolated) leaves of λ and subleaves of the horocycle foliation transverse to λ (assume that λ is maximal), and then expresses the monodromy matrix of α as an infinite product of Mobius transformations

$$\prod N_i.$$

Here, N_i is a Mobius transformation which translates along the imaginary axis or the horocycle leaves. Here, for each i, the translation N_i varies real analytically in t. Thurston proves the analyticity by showing that a specific sequence of finite product approximations of $\prod N_i$ converges uniformly (see [72, Chapter 2, §3.5]) for a detailed calculation).

In our setting, we replace the horocycle foliation by the foliation η which is the extension to Y of the pushforward of the vertical foliation of the Hopf differential of $f: X \to Y \setminus \lambda$ (Lemma 7.7), and express the monodromy of α as infinite product of Mobius transformations

$$\prod M'_i,$$

where M'_i is a Mobius transformation that translates along the imaginary axis or the leaves of η. Here, Lemma 7.4 guarantees the analyticity of M'_i in $t>0$. Applying the same computation as in [90, §4] (see also [72, Chapter 2, §3.5]) allows us to conclude that the sequence of finite product approximations of $\prod M'_i$ converges uniformly. In particular, the limit $\prod M'_i$ also varies real analytically in t.

We now elaborate on the details. Let η be the extension to Y of the pushforward of the vertical foliation of the Hopf differential of $f: X \to Y \setminus \lambda$ (Lemma 7.7). Consider the ε-neighbourhood of λ. There exists a realization α^* of the simple closed curve α which is the concatenation of finitely many subarcs of leaves bounding complementary regions of λ and finitely many subarcs of leaves of η. Fix an orientation of α^*. We may relabel those subarcs in a cyclic order as $\{\gamma_i\}$ according to the orientation of α^*. Next, we associate a matrix M_i to each of γ_i , and write the monodromy of α as a finite product $\prod_{i=1}^m M_i$. In particular, let V be a continuous unit vector field along α^* such that at every endpoint p of γ_i, the vector $V(p)$ is tangent to the leaf of λ that contains p. The orientation of α^* induces an orientation of each arc γ_i. For γ_i, we lift the vector at the starting point to the upward vertical unit vector at $i\in\mathbb{H}^2$. Let M_i be the Mobius transformation which moves the lifted vector at i to the lifted vector at the other endpoint of the lift of γ_i. The monodromy of α^* is the product $\prod_{i=1}^m M_i$.

To show that $\prod_{i=1}^m M_i$ is real analytic in t, it suffices to show that each M_i is real analytic in t. If the subarc γ_i corresponding to M_i is contained in a (boundary) leaf of λ, then, by the construction/definition of M_i in the proceeding paragraph, the Mobius transformation M_i moves $i\in\mathbb{H}^2$ along the imaginary axis by the distance $t\ell_Y(\gamma_i)$, and hence is real analytic. We now consider the case where γ_i is a subarc contained in some leaf of η. As we mentioned earlier, we will express this M_i as an infinite product of Mobius transformations corresponding to the subarcs of $\gamma_i\setminus\lambda$, show that each term in the product is real analytic in t, and prove that a specific sequence of finite approximations converges uniformly.

Before that, we need some estimates on lengths of $\gamma_i\setminus\lambda$ leading to estimates on the Mobius transformations M_i (here and in the next three paragraphs). Notice that, for any fixed $\varepsilon'>0$, the number of components of $\gamma_i\setminus\lambda$ with length at least ε' is finite. In particular, there are only finitely many components of $\gamma_i\setminus\lambda$ not contained in the cusp region of $Y_t\setminus\lambda$. Let $\wedge$ be an arbitrary cusp region of $Y\setminus\lambda$ with $\lambda_0\subset\lambda$ being one of the bounding ideal geodesics of $\wedge$. Let q_0 be a point on λ_0. Let $\{q_{i_n}\}$ be the endpoints of $\gamma_i\setminus\lambda$ on λ_0, labelled in an increasing way according to their distances to the base point q_0 along λ_0. We label the components of $\gamma_i\setminus\lambda$ in $\wedge$ by $\widehat{\gamma}_{i_n}$ such that the endpoint of $\widehat{\gamma}_{i_n}$ on λ_0 is q_{i_n}.

Let $[a,b]\subset(0,\infty)$ be an arbitrary closed interval of finite length on which we will prove analyticity in t. Let r_0 be the infimum of the length of the shortest simple closed curves on Y_t as t varies in $[a,b]$. Since Y_t varies continuously in t on this compact set $[a,b]$, we see that $r_0>0$. After dividing γ_i into smaller arcs if necessary, we may assume that the length of γ_i is much smaller than $\frac{1}{2}r_0$. This implies there exists some constant $r>\frac{1}{2}r_0$ such that the hyperbolic distance on Y_a between any pair of consecutive points of $\{q_{i_n}\}$ is at least r (or else we would find a closed curve comprising a path in λ_0 and γ_i of length less than r_0). To simplify the notation, since our attention is focused on a single arc γ_i, we can set $q_{i_n}=q_n$.

Since the induced map $Y_a\to Y_t$ expands by a factor of $\sqrt{t/a}$ along λ, the distance

$d_t(q_0, q_n)$ along λ_0 on Y_t satisfies:

$$d_t(q_0, q_n) = \sqrt{\frac{t}{a}}\, d_a(q_0, q_n) \geqslant \sqrt{\frac{t}{a}}\, nr.$$

Combining this with (3.3) and Lemma 3.5, we see that the preimage of $\widehat{\gamma}_{i_n}$ on X under the map $f_t\colon X \to Y_t \setminus \lambda$ has $|\mathrm{Hopf}(f_t)|$-distance at least $Cnr\sqrt{t/a}$ from the zeros of $\mathrm{Hopf}(f_t)$ for some constant C which is independent of $t \in [a, b]$: here the point is this. The arc $\widehat{\gamma}_{i_n}$ has pre-image which is a bi-infinite vertical leaf in two vertical half-planes (possibly with some bi-infinite strips of finite height in between) which are being mapped to the cusp — and since $\widehat{\gamma}_{i_n}$ is located at least at hyperbolic distance $Cnr\sqrt{t/a}$ into the cusp, it has preimage at least at a proportional distance in the $|\mathrm{Hopf}(f_t)|$-distance from some base point, such as the zeros of the differential. Again by (3.3) and Lemma 3.5, we see that the hyperbolic length of $\widehat{\gamma}_{i_n}$ on Y_t satisfies

$$\ell(\widehat{\gamma}_{i_n}) \leqslant Ce^{-Cnr\sqrt{t/a}} \tag{7.9}$$

for some positive constant C independent of $t \in [a, b]$.

Fix a base point $x_0 \in \mathbb{H}$ in the hyperbolic plane. Let $\mathbf{d}$ be the complete, left invariant metric on $\mathrm{PSL}(2, \mathbb{R})$ defined as (cf. [72])

$$\mathbf{d}(A, B) := \sup_{x \in \mathbb{H}} d_{\mathbb{H}}(Ax, Bx) e^{-d_{\mathbb{H}}(x, x_0)} \quad \text{for all } A, B \in \mathrm{PSL}(2, \mathbb{R}).$$

For any $A \in \mathrm{PSL}(2, \mathbb{R})$, define $\|A\| := \mathbf{d}(\mathrm{Id}, A)$. Let $\widehat{M}_{i_n}$ be the Mobius transformation corresponding to $\widehat{\gamma}_{i_n}$ defined similarly as M_i. By (7.9), we see that there exists another constant C', independent of $t \in [a, b]$, such that $\|\widehat{M}_{i_n}\| \leqslant C' e^{-Cnr\sqrt{t/a}}$. In particular, there exist two positive constants C_1 and C_2 such that, for any $t \in [a, b]$ and any $n \geqslant 1$,

$$\|\widehat{M}_{i_n}\| \leqslant C_1 e^{-C_2 n}. \tag{7.10}$$

From this, we infer that the distance to $\mathrm{Id} \in \mathrm{PSL}(2, \mathbb{R})$ from any finite product of matrices in $\{\widehat{M}_{i_n}\}$ is bounded from above by

$$C_1 \sum_{n \geqslant 1} e^{-C_2 n} = C_1 (1 - e^{-C_2})^{-1}.$$

We now return to the construction of M_i, the monodromy corresponding to γ_i as an infinite product. (Recall that $\{\widehat{\gamma}_{i_n}\}$ are the components of $\gamma_i \setminus \lambda$, here relaxing the notation to allow the components $\{\widehat{\gamma}_{i_n}\}$ to be subsets of any one of the finitely many cusp regions. Because there are only finitely many cusp regions, the estimates of the previous paragraphs continue to hold.) For each $m \geqslant 1$, consider the (finite) subset I_m of $\{\widehat{\gamma}_{i_n}\}$ which comprises those $\widehat{\gamma}_j$ whose length is at least $1/m$. let $\mathbb{M}_m$ be the product of those $\widehat{M}_{i_n}$ corresponding to $\widehat{\gamma}_{i_n} \in I_m$, according to the order inherited from the orientation of γ_i (recall that $\widehat{\gamma}_{i_n} \subset \gamma_i$). In particular, $\mathbb{M}_{m+1}$ is obtained from $\mathbb{M}_m$ by inserting finitely many $\widehat{M}_{i_n}$ corresponding to $\widehat{\gamma}_{i_n} \in I_{m+1}$. Applying a standard calculation using (7.10) as in [72, pp. 161–162, proof of Proposition 3.15], we see that $\mathbb{M}_m$ converges uniformly in $t \in [a,b]$, where the limit is exactly the Mobius transformation M_i corresponding to γ_i (recall that $\{\widehat{\gamma}_{i_n}\}$ are the components of $\gamma_i \setminus \lambda$). For each $\widehat{\gamma}_{i_n}$, there is exactly one $j_n \in \{1, 2, ..., k\}$ such that the segment $\widehat{\gamma}_{i_n}$ is contained in the component $Y_t^{j_n}$ of $Y_t \setminus \lambda$.

By Lemma 7.4, both $Y_t^{j_n}$ and the realization $f_t^{j_n} \circ (f_1^{j_n})^{-1}(\widehat{\gamma}_{i_n})$ of $\widehat{\gamma}_{i_n}$ on $Y_t^{j_n}$ are both real analytic in t. Therefore, the Mobius transformation $\widehat{M}_{i_n}$ corresponding to $\widehat{\gamma}_{i_n}$ is real analytic in t. Combined with the uniform convergence, this proves that M_i, as the limit of $\mathbb{M}_m$, is analytic in $t \in (a,b)$. Consequently, the monodromy of α, as a product of finitely many M_i, varies analytically in t. Since Y_t is determined by finitely many such α, this implies that Y_t is analytic in $t \in (a,b)$. The arbitrariness of $[a,b] \subset (0,\infty)$ then implies that Y_t varies analytically in $t \in (0,\infty)$. □

Definition 7.9. Given a closed hyperbolic surface $Y \in \mathcal{T}(S)$, a geodesic lamination λ on Y, and a harmonic diffeomorphism $f: X \to Y \setminus \lambda$ from some (possibly disconnected) punctured Riemann surface. The path

$$\mathrm{PSL}_{Y,\lambda,f}: (0,\infty) \longrightarrow \mathcal{T}(S)$$

constructed in Theorem 1.5 is called a *piecewise harmonic stretch line* defined by the triple (Y, λ, f). The restriction to $[1,\infty)$ is called a *piecewise harmonic stretch ray* defined by (Y, λ, f), denoted by $\mathbf{PSR}_{Y,\lambda,f}: [1,\infty) \to \mathcal{T}(S)$.

From the construction, we see that $\mathrm{PSL}_{Y,\lambda,f}(1) = Y$. In particular, the piecewise harmonic stretch ray $\mathbf{PSR}_{Y,\lambda,f}$ starts at Y. Each piecewise harmonic stretch line/ray admits a canonical orientation coming from the orientation of the positive real ray $\{t > 0\}$. In that orientation, a piecewise harmonic stretch line is a (reparameterized) geodesic in the Thurston metric (Theorem 1.5 (b)).

We end this subsection with the following identification.

LEMMA 7.10. *Let $Y \in \mathcal{T}(S)$ and λ be a maximal geodesic lamination λ. Then, for any harmonic diffeomorphism $f: X \to Y \setminus \lambda$ from a punctured Riemann surface X, the piecewise harmonic stretch line* $\mathrm{PSL}_{Y,\lambda,f}$ *and the Thurston stretch line* $\mathbf{SL}_{Y,\lambda}$ (*see* §2.6 *for definition*) *coincide.*

Proof. We begin with recalling the construction of the Thurston stretch line $\mathbf{SL}_{Y,\lambda}$. Let $Y^1, ..., Y^{4g-4}$ be the set of ideal triangles complementary to λ on Y. Let F be the horocyclic partial foliation of λ on Y. On each component Y^i, the horocycle partial foliation $\widehat{F}^i:=F|_{Y^i}$ foliates Y^i except for a central region bounded by three pairwise tangent horocyclic arcs meeting each boundary edge of Y^i perpendicularly. Let F^i be a measured foliation obtained from $\widehat{F}^i$ by pushing the leaves of $\widehat{F}^i$ toward the central region of Y^i while keeping endpoints on ∂Y invariant and keeping the resulting leaves orthogonal to ∂Y^i. Note that any two thus obtained F^i's differ by an isotopy of Y^i fixing ∂Y^i pointwise. The transverse measure on F^i is defined as follows. For any arc τ on Y^i that is transverse to F^i, there exists one boundary geodesic of Y^i, to which we may project τ along leaves of F^i. The transverse measure on τ is then defined to be the hyperbolic length of the projection of τ to that geodesic (which is independent of the choice of geodesics).

Let $X^1, ..., X^{4g-4}$ be the corresponding components of X and $f^i: X^i \to Y^i$ be the restriction of f to X^i. Note that, for each i, the Riemann surface X^i is conformally the complex plane. That Y^i is an ideal triangle implies that the Hopf differential Φ^i of f^i is a linear polynomial, that is, $\Phi^i = a^i(z-b^i)dz^2$ for some pair of complex numbers (a^i, b^i) (item (i) of Lemma 5.9). The critical graph of the horizontal measured foliation of Φ^i is a tripod, whose complement consists of three half-planes. In particular, the horizontal foliation is symmetric about each of the three half-infinite critical leaves. On the other hand, each ideal triangle is symmetric about any of the three half-infinite geodesic rays $\{h^{i1}, h^{i2}, h^{i3}\}$, where each starts perpendicularly from a boundary edge and converges to the opposite ideal point. Applying Theorem 7.1 to the triple (X^i, Y^i, Φ^i) with respect the principal part given by Φ^i, we see that the harmonic diffeomorphism $f^i: X^i \to Y^i$ is symmetric about each of $\{h^{i1}, h^{i2}, h^{i3}\}$. Accordingly, the pushforward V^i of the vertical foliation of Φ^i is also symmetric about each of $\{h^{i1}, h^{i2}, h^{i3}\}$. By Lemma 7.8, the transverse measure on V^i coincides with half of the hyperbolic length of the projection to boundary geodesics of Y^i. More precisely, for any arc τ on Y^i that is transverse to V^i, there exists one boundary geodesic of Y^i, to which we may project τ along leaves of V^i. By Lemma 7.8, the transverse measure on τ equals half of the hyperbolic length of the projection of τ to that geodesic (which is independent of the choice of geodesics).

Combining the discussion in the preceding two paragraphs, we see that $2V^i$ and F^i differ by an isotopy of Y^i fixing ∂Y^i pointwise. This also holds for $2tV^i$ and tF_i, for any $t>0$. Therefore, the double of the extended foliation tV obtained from $tV^1, ..., tV^{4g-4}$ via Lemma 7.7 also differs from tF by an isotopy of Y that fixes $\bigcup_i \partial Y^i$ pointwise. This implies that for each $t>0$, the surface $\mathrm{PSL}_{X,\lambda,f}(t)$ and $\mathbf{SL}_{Y,\lambda}(t)$ coincide since both of them are obtained from gluing ideal triangles with the same transverse measured foliation $tF = 2tV$. $\square$

7.6. Limits of piecewise harmonic stretch lines

Recall that every piecewise harmonic stretch line is directed. We next prove a proposition that will be useful when we define canonical harmonic stretch lines from given base points to points on the Thurston boundary of Teichmüller space in §13.

PROPOSITION 7.11. *Let $Y\in\mathcal{T}(S)$ be a closed hyperbolic surface and λ a geodesic lamination on Y. Let $f: X\to Y\setminus\lambda$ be a harmonic diffeomorphism from some punctured surface X. Let β be the pushforward to Y of the vertical foliation of* $\mathrm{Hopf}(f)$. *Then, the piecewise harmonic stretch line determined by (Y,λ,X,f) converges to $[\beta]\in\mathcal{PML}(S)$ in the Thurston compactification in the forward direction.*

Proof. Let PSL be the piecewise harmonic stretch line determined by (Y,λ,X,f). Let $\widehat{Y}_t$ be the (possibly disconnected) crowned hyperbolic surface determined by the pair $(X, t\,\mathrm{Hopf}(f))$ by Theorem 7.1. Let $Y_t\in\mathrm{PSL}$ be the hyperbolic surface such that $Y_t\setminus\lambda=\widehat{Y}_t$. Let α be an arbitrary simple closed curve on Y. Consider the pushforward to Y of the horizontal and vertical foliations of $\mathrm{Hopf}(f)$. By Lemma 7.7, the pushforward of the vertical foliation extends to a unique measured foliation on the whole surface Y_t. For any $\varepsilon>0$, let α^* be a representative of α satisfying the following properties:

- α^* consists of segments of the pushforward of horizontal and vertical foliations of $\mathrm{Hopf}(f)$ alternatively;
- α^* avoids the zeros of $\mathrm{Hopf}(f)$;
- α^* almost realizes the (minimal) intersection number with β, the vertical foliation of $\mathrm{Hopf}(f)$, namely

$$\left|\int_{\alpha^*}\left|\mathrm{Re}\sqrt{\mathrm{Hopf}(f)}\right|-i(\alpha,\beta)\right|<\varepsilon. \tag{7.11}$$

To see such a α^* exists, let α' be a representative of α which realizes the minimal intersection number with β, that is,

$$\int_{\alpha^*}\left|\mathrm{Re}\sqrt{\mathrm{Hopf}(f)}\right|=i(\alpha,\beta).$$

Next, we deform α' to get a new representative α'' that consists of segments of the pushforward of horizontal and vertical foliations of $\mathrm{Hopf}(f)$ alternatively. Then, α'' also realizes the minimal intersection number with β. Finally, at each zero of $\mathrm{Hopf}(f)$ on X, we pick a polygon Q consisting of horizontal and vertical segments alternatively each of which is of distance $\varepsilon'\ll\varepsilon$ to the enclosed zero under the flat metric induced by the Hopf differential $\mathrm{Hopf}(f)$, and then push the subsegments of α'' inside the image $f(Q)$ of this polygon to the boundary of $f(Q)$; here the choice of ε' depends on α. This gives the desired representative α^*.

Let $d>0$ be the distance of $f^{-1}(\alpha^*)$ to the zeros of $\mathrm{Hopf}(f)$. Let K_v be the number of vertical segments of α^*. Then, by (3.3) and Lemma 3.5, for $t>0$ sufficiently large, the total length $\mathrm{Length}_{Y_t}({}^v\alpha^*)$ of vertical segments of α^* with respect to the hyperbolic metric $Y_t\in\mathrm{PSL}$ satisfies

$$\mathrm{Length}_{Y_t}({}^v\alpha^*)\leqslant K_v\int_{td}^{\infty}2\sinh^{-1}\left(\frac{\chi(Y)}{s^2}\right)\exp(-s)\,ds\to 0,\quad \text{as } t\to+\infty,$$

and the total length $\mathrm{Length}_{Y_t}({}^h\alpha^*)$ of horizontal segments of α^* with respect to the hyperbolic metric $Y_t\in\mathrm{PSL}$ satisfies

$$\begin{aligned}\mathrm{Length}_{Y_t}({}^h\alpha^*)&\leqslant 2\int_{\alpha^*}\left|\mathrm{Re}\sqrt{t\,\mathrm{Hopf}(f)}\right|\left(1+\exp(-td)\sinh^{-1}\left(\frac{\chi(Y)}{(td)^2}\right)\right)\\ &\to 2\sqrt{t}\int_{\alpha^*}\left|\mathrm{Re}\sqrt{\mathrm{Hopf}(f)}\right|,\quad \text{as } t\to+\infty.\end{aligned}$$

Therefore, the length $\mathrm{Length}_{Y_t}(\alpha^*)$ of α^* on Y_t satisfies

$$\begin{aligned}\limsup_{t\to+\infty}\frac{\mathrm{Length}_{Y_t}(\alpha^*)}{\sqrt{t}}&=\limsup_{t\to+\infty}\frac{\mathrm{Length}_{Y_t}({}^v\alpha^*)+\mathrm{Length}_{Y_t}({}^h\alpha^*)}{\sqrt{t}}\\ &\leqslant 2\int_{\alpha^*}\left|\mathrm{Re}\sqrt{\mathrm{Hopf}(f)}\right|.\end{aligned}$$

Hence, the length of the geodesic representative of α on Y_t satisfies

$$\limsup_{t\to+\infty}\frac{\ell_{Y_t}(\alpha)}{\sqrt{t}}\leqslant\limsup_{t\to+\infty}\frac{\mathrm{Length}_{Y_t}(\alpha^*)}{\sqrt{t}}\leqslant 2\int_{\alpha^*}\left|\mathrm{Re}\sqrt{\mathrm{Hopf}(f)}\right|. \tag{7.12}$$

On the other hand, by (3.3),

$$\ell_{Y_t}(\alpha)\geqslant 2\int_{\alpha^*}\left|\mathrm{Re}\sqrt{t\,\mathrm{Hopf}(f)}\right|=2\sqrt{t}\int_{\alpha^*}\left|\mathrm{Re}\sqrt{\mathrm{Hopf}(f)}\right|.$$

Combined with (7.12), this implies that

$$\lim_{t\to+\infty}\frac{\ell_{Y_t}(\alpha)}{\sqrt{t}}=2\int_{\alpha^*}|\,\mathrm{Re}\sqrt{\mathrm{Hopf}(f)}|.$$

It then follows from (7.11) and the arbitrariness of ε that

$$\lim_{t\to+\infty}\frac{\ell_{Y_t}(\alpha)}{\sqrt{t}}=2i(\alpha,\beta).$$

This proves that Y_t converges to $[\beta]\in\mathcal{PML}(S)$ as $t\to+\infty$. □

7.7. Generalized maximum principle

To prove Lemma 7.3, we need the following generalized maximum principle from [13].

THEOREM 7.12. ([13, Theorems 3 and 8]) *Let M be a complete non-compact manifold with Ricci curvature bounded from below by $-K$ for some constant $K \geqslant 0$. Let u be a C^2 function on M.*

(I) *If u is bounded from above, then there exists a sequence of points $p_k \in M$ such that*

$$\lim_{k\to\infty} u(p_k) = \sup_{p\in M} u(p), \quad \lim_{k\to\infty} |\nabla u(p_k)| = 0 \quad \textit{and} \quad \limsup_{k\to\infty} \Delta u(p_k) \leqslant 0.$$

(II) *If u satisfies the differential inequality $\Delta u \geqslant f(u)$, where f is a function on $\mathbb{R}$ with the property that there exists a continuous non-decreasing function g positive on some interval $[a,\infty)$ such that*

$$\liminf_{t\to\infty} \frac{f(t)}{g(t)} > 0$$

and

$$\int_b^\infty \left(\int_a^t g(\tau)\, d\tau \right)^{-1/2} dt < \infty \quad \textit{for some } b \geqslant a,$$

then u is bounded from above. Furthermore, if f is lower semi-continuous, then

$$f(\sup u) \leqslant 0.$$

7.8. Geometry of harmonic map rays from degenerated surfaces

To prove Lemma 7.3, we need a generalization of [96, Proposition 4.3]. Before that, we need to recall some notation. Let $(M, \sigma|dz|^2)$ be the hyperbolic plane or the complex plane and $(\mathbb{H}^2, \rho|dw|^2)$ the hyperbolic plane. Let $f_t: M \to (\mathbb{H}^2, \rho|dw|^2)$ be a harmonic diffeomorphism with Hopf differential $t\Phi$. Recall the conformal energy density

$$\mathcal{H}_t = \frac{\rho(f_t)}{\sigma} \left| \frac{\partial f_t}{\partial z} \right|^2,$$

the anti-conformal energy density

$$\mathcal{L}_t = \frac{\rho(f_t)}{\sigma} \left| \frac{\partial f_t}{\partial \bar{z}} \right|^2,$$

the Laplacian operator

$$\Delta_\sigma = \frac{4}{\sigma} \frac{\partial^2}{\partial z \partial \bar{z}},$$

the Beltrami differential

$$\nu_t = \frac{(\partial f_t/\partial \bar z)\, d\bar z}{(\partial f_t/\partial z)\, dz}$$

and identities (3.5)–(3.11) relating them.

PROPOSITION 7.13. *Let $(M,\sigma|dz|^2)$ be the hyperbolic plane $\mathbb{H}^2$ or the complex plane $\mathbb{C}$. Let Φdz^2 be a holomorphic quadratic differential on M; in the case where M is the complex plane, we assume that Φ is a polynomial. Let $f_t\colon (M,\sigma|dz|^2)\to\mathbb{H}^2$ be a family of harmonic diffeomorphisms onto the corresponding images with Hopf differentials $\Phi_t=t\Phi$, where $t>0$. Then, the following statements hold, where we use primes to indicate differentiation with respect to t.*

(i) *For any $p\in M$, $\mathcal{H}_t'(p)\geqslant 0$.*

(ii) *For any $p\in M$, $\mathcal{L}_t'(p)\geqslant 0$.*

(iii) *For any $p\in M$, $|\nu_t|'(p)\geqslant 0$, and $|\nu_t|'(p)>0$ if $\Phi(p)\neq 0$.*

(iv) *The function*

$$\frac{|\nu_t|'}{|\nu_t|}(p)$$

extends to a bounded positive analytic function on M. In particular, for any compact subset N of M, there exists a constant $\varepsilon>0$ such that

$$\frac{|\nu_t|'}{|\nu_t|}(p) > \varepsilon \quad \text{for any } p\in N.$$

Proof. The idea is to use the maximum principle and the generalized maximum principle.

Notice that, for any fixed $t\geqslant 0$,

$$\inf_{p\in M} \mathcal{H}_t(p) > 0. \tag{7.13}$$

In fact, if M is the hyperbolic plane, then it follows from [94, Theorem 12 and Proposition 10] that $\mathcal{H}_t(p)\geqslant 1$ for all $t\geqslant 0$ and all $p\in M$. If M is the complex plane, since Φ is a non-constant polynomial, then by (3.6), since $\mathcal{H}_t(p)\geqslant\mathcal{L}_t(p)$ (notice f_t is orientation-preserving), we have that $\mathcal{H}_t(p)\geqslant t|\Phi(p)|\to\infty$, as $p\to\infty$. In particular, there exists $R>0$ such that $\mathcal{H}_t(p)>1$ for all $|p|>R$. On the other hand, on $\{p\in\mathbb{C}: |p|\leqslant R\}$, we have that $\mathcal{H}_t(p)$ is positive and continuous. Therefore, in both cases, (7.13) holds.

First, we show that $\mathcal{H}_t'(p)\geqslant 0$ for all $p\in M$. To this end, it is equivalent to show that, for any fixed $p\in M$, $\mathcal{H}_t(p)$ is an increasing function of $t>0$. For any $0<t<s$, let $\mathcal{W}_t=\frac{1}{2}\log\mathcal{H}_t$. Then, by (3.9), we have the Bochner-type equations

$$\begin{aligned}\Delta_\sigma \mathcal{W}_t &= e^{2\mathcal{W}_t} - \frac{t^2|\Phi|^2}{\sigma^2} e^{-2\mathcal{W}_t} + K(\sigma), \\ \Delta_\sigma \mathcal{W}_s &= e^{2\mathcal{W}_s} - \frac{s^2|\Phi|^2}{\sigma^2} e^{-2\mathcal{W}_s} + K(\sigma).\end{aligned} \tag{7.14}$$

By subtracting the two equations, we get

$$\Delta_\sigma(\mathcal{W}_t-\mathcal{W}_s)=(e^{2\mathcal{W}_t}-e^{2\mathcal{W}_s})-\frac{t^2|\Phi|^2}{\sigma^2}e^{-2\mathcal{W}_t}+\frac{s^2|\Phi|^2}{\sigma^2}e^{-2\mathcal{W}_s}.$$

Let $\tilde{\sigma}=\tilde{\sigma}(s,z):=\sigma e^{2\mathcal{W}_s}|dz|^2$ and $\eta:=\mathcal{W}_t-\mathcal{W}_s$. Then,

$$\begin{aligned}\Delta_{\tilde{\sigma}}\eta &= e^{2\eta}-1-\frac{t^2}{s^2}|\nu_s|^2e^{-2\eta}+|\nu_s|^2 && \text{(by (3.11))}\\ &\geqslant e^{2\eta}-e^{-2\eta}-1 && \text{(since } |\nu_s|\leqslant 1\text{)}.\end{aligned}\tag{7.15}$$

Recall that $\tilde{\sigma}=\tilde{\sigma}(s)$ is a complete metric with curvature bounded from below: indeed, for fixed s,

$$\begin{aligned}K(\tilde{\sigma}) &= -2\Delta_{\tilde{\sigma}}\log\tilde{\sigma}\\ &= -\frac{2\sigma}{\tilde{\sigma}}\Delta_\sigma(\log\sigma+2\mathcal{W}_s)\\ &= -e^{-2\mathcal{W}_s}(2\Delta_\sigma\log\sigma+4\Delta_\sigma\mathcal{W}_s)\\ &= -e^{-2\mathcal{W}_s}\left(-K(\sigma)+4e^{2\mathcal{W}_s}-4\frac{s^2|\Phi|^2}{\sigma^2}e^{-2\mathcal{W}_s}+4K(\sigma)\right) && \text{(by (3.6) and (3.9))}\\ &= -3K(\sigma)e^{-2\mathcal{W}_s}-4+4|\nu_s|^2 && \text{(by (3.11))}\\ &\geqslant -4 && \text{(since } K(\sigma)\in\{0,-1\}\text{)}.\end{aligned}\tag{7.16}$$

It then follows from Theorem 7.12, by taking $f(t)=e^{2t}-e^{-2t}-1$ and $g(t)=e^{2t}$, that

$$\bar{\eta}:=\sup_{p\in M}\eta(p)<+\infty.$$

Combining this boundedness with item (I) of Theorem 7.12 and the fact that the curvature of $\tilde{\sigma}$ is bounded from below (7.16), we see that there exists a sequence $p_k\in M$, such that

$$\lim_{k\to\infty}\eta(p_k)=\bar{\eta}=\sup_{p\in M}\eta(p)\quad\text{and}\quad\limsup_{k\to\infty}\Delta_{\tilde{\sigma}}\eta(p_k)\leqslant 0.$$

By taking a subsequence if necessary, we may assume that

$$\lim_{k\to\infty}|\nu_s(p_k)|^2=a\in[0,1].$$

It then follows from (7.15) that

$$\begin{aligned}0 &\geqslant \limsup_{k\to\infty}\Delta_{\tilde{\sigma}}\eta(p_k)\\ &= \limsup_{k\to\infty}\left(e^{2\eta(p_k)}-1-\frac{t^2}{s^2}|\nu_s(p_k)|^2e^{-2\eta(p_k)}+|\nu_s(p_k)|^2\right)\\ &= e^{2\bar{\eta}}-\frac{t^2}{s^2}ae^{-2\bar{\eta}}-1+a\\ &\geqslant e^{2\bar{\eta}}-ae^{-2\bar{\eta}}-1+a && \text{(since } 0<t<s\text{)},\end{aligned}$$

which implies that $\bar{\eta}\leqslant 0$. Therefore,

$$\mathcal{W}_t(p)-\mathcal{W}_s(p)=\eta(p)\leqslant\bar{\eta}\leqslant 0$$

Hence, $\mathcal{H}_t(p)\leqslant\mathcal{H}_s(p)$ for all $p\in M$. Equivalently, $\mathcal{H}_t'(p)\geqslant 0$ for all $p\in M$.

Next, we show that $\mathcal{L}_t'(p)\geqslant 0$ for all $p\in M$. Notice that

$$\frac{t^2|\Phi|^2}{\sigma^2}=\mathcal{H}_t\mathcal{L}_t \quad \text{and} \quad \frac{s^2|\Phi|^2}{\sigma^2}=\mathcal{H}_s\mathcal{L}_s.$$

Then,

$$\frac{\mathcal{L}_t}{\mathcal{L}_s}(p)=\frac{t^2}{s^2}\frac{\mathcal{H}_s}{\mathcal{H}_t}(p), \quad \Phi(p)\neq 0 \tag{7.17}$$

extends to a well-defined analytic function on the whole surface M, still denoted by

$$\frac{\mathcal{L}_t}{\mathcal{L}_s}(p)$$

for simplicity. Let

$$\delta(p)=\frac{1}{2}\log\frac{\mathcal{L}_t}{\mathcal{L}_s}(p).$$

The equation above implies that

$$\delta(p)=\log\frac{t}{s}+\frac{1}{2}\log\frac{\mathcal{H}_s}{\mathcal{H}_t}(p).$$

It then follows that

$$\begin{aligned}\Delta_\sigma\delta &= \frac{1}{2}\Delta_\sigma\log\mathcal{H}_s-\frac{1}{2}\Delta_\sigma\log\mathcal{H}_t\\ &= \mathcal{H}_s-\mathcal{H}_t-\frac{s^2|\Phi|^2}{\sigma^2}\mathcal{H}_s^{-1}+\frac{t^2|\Phi|^2}{\sigma^2}\mathcal{H}_t^{-1}.\end{aligned}$$

Let $\hat{\sigma}:=\sigma\mathcal{H}_t|dz|^2$. Then,

$$\begin{aligned}\Delta_{\hat{\sigma}}\delta &= \frac{\mathcal{H}_s}{\mathcal{H}_t}-1-\frac{s^2|\Phi|^2}{\sigma^2\mathcal{H}_s\mathcal{H}_t}+\frac{t^2|\Phi|^2}{\sigma^2\mathcal{H}_t^2}\\ &= \frac{\mathcal{H}_s}{\mathcal{H}_t}-1-\frac{t^2|\Phi|^2}{\sigma^2\mathcal{H}_t^2}\frac{s^2}{t^2}\frac{\mathcal{H}_t}{\mathcal{H}_s}+\frac{t^2|\Phi|^2}{\sigma^2\mathcal{H}_t^2}\\ &= \frac{\mathcal{H}_s}{\mathcal{H}_t}-1-|\nu_t|^2\frac{s^2}{t^2}\frac{\mathcal{H}_t}{\mathcal{H}_s}+|\nu_t|^2 && \text{(by (3.11))}\\ &= \frac{s^2}{t^2}e^{2\delta}-1-|\nu_t|^2e^{-2\delta}+|\nu_t|^2 && \text{(by (7.17))}\\ &\geqslant \frac{s^2}{t^2}e^{2\delta}-1-e^{-2\delta}.\end{aligned}$$

By applying an argument similar to the one for the case of η, we see that $\sup_{p\in M}\delta<0$. In particular, $\mathcal{L}_t<\mathcal{L}_s$ for all $0\leqslant t<s$ and all $p\in M$ with $\Phi(p)\neq 0$. This implies that $\mathcal{L}_t'(p)\geqslant 0$ for all $t\geqslant 0$ and all $p\in M$ with $\Phi(p)\neq 0$. It then follows from continuity of $\mathcal{L}_t'$ that $\mathcal{L}_t'(p)\geqslant 0$ for all $t\geqslant 0$ and all $p\in M$.

It remains to show items (iii) and (iv). Suppose that $\Phi(p)\neq 0$. Taking derivatives respect to t for (3.6) and (3.11), we see that

$$\frac{\mathcal{H}_t'}{\mathcal{H}_t}(p)+\frac{\mathcal{L}_t'}{\mathcal{L}_t}(p)=\frac{2}{t}, \tag{7.18}$$

$$\frac{\mathcal{H}_t'}{\mathcal{H}_t}(p)+\frac{|\nu|_t{}'}{|\nu_t|}(p)=\frac{1}{t}. \tag{7.19}$$

Therefore, for any fixed $t>0$, both $\mathcal{L}_t'/\mathcal{L}_t$ and $|\nu|_t'/|\nu|_t$ extend to be bounded non-negative analytic functions on the whole of M. We next improve this bound on $\mathcal{L}_t'/\mathcal{L}_t$.

Applying Theorem 7.12 to $\mathcal{H}_t'/\mathcal{H}_t$, which is a bounded non-negative analytic function by (7.18), we see that there exists a sequence of points $p_k\in M$ with $p_k\to\infty$, such that

$$\lim_{k\to\infty}\frac{\mathcal{H}_t'}{\mathcal{H}_t}(p_k)=\sup_{p\in M}\frac{\mathcal{H}_t'}{\mathcal{H}_t}(p) \quad\text{and}\quad \limsup_{k\to\infty}\Delta_\sigma\frac{\mathcal{H}_t'}{\mathcal{H}_t}(p_k)\leqslant 0. \tag{7.20}$$

Therefore,

$$\begin{aligned}
2\sup_{p\in M}\frac{\mathcal{H}_t'}{\mathcal{H}_t}(p)-\frac{2}{t}&=\lim_{k\to\infty}\left(2\frac{\mathcal{H}_t'}{\mathcal{H}_t'}(p_k)-\frac{2}{t}\right)\\
&=\lim_{k\to\infty}\left(\frac{\mathcal{H}_t'}{\mathcal{H}_t}(p_k)-\frac{\mathcal{L}_t'}{\mathcal{L}_t}(p_k)\right)\\
&\leqslant\lim_{k\to\infty}\left(\frac{\mathcal{H}_t'}{\mathcal{H}_t}(p_k)-\frac{\mathcal{L}_t'}{\mathcal{H}_t}(p_k)\right)\quad(\text{since } \mathcal{H}_t>\mathcal{L}_t\geqslant 0 \text{ and } \mathcal{L}_t'\geqslant 0)\\
&=\lim_{k\to\infty}\frac{1}{\mathcal{H}_t(p_k)}\frac{1}{2}\Delta_\sigma\frac{\mathcal{H}_t'}{\mathcal{H}_t}(p_k)\quad(\text{by } (3.9))\\
&\leqslant 0 \quad(\text{by } (7.13) \text{ and } (7.20)).
\end{aligned}$$

Consequently,

$$\sup_{p\in M}\frac{\mathcal{H}_t'}{\mathcal{H}_t}(p)\leqslant\frac{1}{t}. \tag{7.21}$$

Next, we claim that

$$\frac{\mathcal{H}_t'}{\mathcal{H}_t}(p)<\frac{1}{t} \tag{7.22}$$

holds for any $p\in M$. Suppose to the contrary that there exists some $p_0\in M$ such that

$$\frac{\mathcal{H}_t'}{\mathcal{H}_t}(p_0)=\frac{1}{t}.$$

Combined with (7.21), this implies that

$$\Delta_\sigma \frac{\mathcal{H}_t'}{\mathcal{H}_t}(p_0) \leqslant 0.$$

It then follows from (3.9) that

$$\mathcal{H}_t'(p_0) - \mathcal{L}_t'(p_0) = \Delta_\sigma \frac{\mathcal{H}_t'}{\mathcal{H}_t}(p_0) \leqslant 0,$$

which implies $\mathcal{H}_t'(p_0) \leqslant \mathcal{L}_t'(p_0)$. Combined with (7.18) and the fact that $\mathcal{H}_t(p_0) > \mathcal{L}_t(p_0)$, this yields that

$$\frac{\mathcal{H}_t'}{\mathcal{H}_t}(p_0) < \frac{1}{2}\left(\frac{\mathcal{H}_t'}{\mathcal{H}_t}(p_0) + \frac{\mathcal{L}_t'}{\mathcal{L}_t}(p_0)\right) = \frac{1}{t}.$$

This contradicts the assumption that

$$\frac{\mathcal{H}_t'}{\mathcal{H}_t}(p_0) = \frac{1}{t}.$$

Hence,

$$\frac{\mathcal{H}_t'}{\mathcal{H}_t}(p) < \frac{1}{t} \quad \text{for all } p \in M \text{ and all } t > 0.$$

It then follows from (7.19) that

$$\frac{|\nu_t|'}{|\nu_t|}(p) = \frac{1}{t} - \frac{\mathcal{H}_t'}{\mathcal{H}_t}(p) > 0 \quad \text{for all } p \in M \text{ and all } t > 0.$$

Combined with the continuity of $|\nu_t|'/|\nu_t|$, this proves (iii) and (iv). □

Proof of Lemma 7.3. Let $p \in X \setminus D$ be an arbitrary point with $\Phi(p) \neq 0$. By the third item of Proposition 7.13, we see that $|\nu_t|_p$ is a strictly increasing function of $t \in (0, \infty)$. Therefore, the function $\mathscr{G}(p,t) := \log(1/|\nu_t(p)|)$ is a strictly decreasing function of $t \in [0, \infty)$ at p with $\Phi(p) \neq 0$. Recall that with respect to the canonical coordinate charts $z = x + iy$ of Φ near p, the pullback metric of Y_t by f_t is

$$2t(\cosh \mathscr{G}(p,t) + 1)\, dx^2 + 2t(\cosh \mathscr{G}(p,t) - 1)\, dy^2. \tag{7.23}$$

Then, for any $s > t > 0$, the Lipschitz constant of $f_s \circ (f_t)^{-1}$ at $f_t(p)$ satisfies

$$\operatorname{Lip}(f_s \circ (f_t)^{-1})|_{f_t(p)} \leqslant \max\left\{ \sqrt{\frac{2s(\cosh \mathscr{G}(p,s)+1)}{2t(\cosh \mathscr{G}(p,t)+1)}}, \sqrt{\frac{2s(\cosh \mathscr{G}(p,s)-1)}{2t(\cosh \mathscr{G}(p,t)-1)}} \right\} < \sqrt{\frac{s}{t}}.$$

Combined with Lemma 3.5, this implies that, as p approaches D (the set of punctures of $X \setminus D$), the Lipschitz constant $\operatorname{Lip}(f_s \circ (f_t)^{-1})|_{f_t(p)}$ converges uniformly to $\sqrt{s/t}$.

We now consider Lipschitz constants of $f_s\circ f_t^{-1}$ at zeros of Φ. Let p_0 be an arbitrary zero of Φ. Then, by (3.7), the value $\nu_t(p_0)=0$ for all $t\geqslant 0$. Hence,

$$\mathscr{G}(p_0,t)=+\infty \quad\text{and}\quad \lim_{p\to p_0}\mathscr{G}(p,t)=+\infty. \tag{7.24}$$

Let U be a small neighbourhood of p_0. By item (iv) of Proposition 7.13, for any $0<t<s$, there exists $\varepsilon>0$ such that

$$\frac{|\nu_r|'}{|\nu_r|}(p)>\varepsilon$$

for any $p\in U$ and any $r\in[t,s]$. It then follows, by integrating this inequality, that

$$\frac{|\nu_s|}{|\nu_t|}(p)=\exp(\log|\nu_s|(p)-\log|\nu_t|(p))>\exp(\varepsilon(s-t)) \tag{7.25}$$

holds for any $p\in U$. Therefore, the Lipschitz constant of $f_s\circ(f_t)^{-1}$ at $f_t(p_0)$ satisfies

$$\begin{aligned}
&\mathrm{Lip}(f_s\circ(f_t)^{-1})|_{f_t(p_0)}\\
&\quad=\lim_{p\to p_0}\mathrm{Lip}(f_s\circ(f_t)^{-1})|_{f_t(p)}\\
&\quad\leqslant\lim_{p\to p_0}\max\left\{\sqrt{\frac{2s(\cosh\mathscr{G}(p,s)+1)}{2t(\cosh\mathscr{G}(p,t)+1)}},\sqrt{\frac{2s(\cosh\mathscr{G}(p,s)-1)}{2t(\cosh\mathscr{G}(p,t)-1)}}\right\}\\
&\quad=\lim_{p\to p_0}\sqrt{\frac{s}{t}\frac{1/|\nu_s|(p)}{1/|\nu_t|(p)}} &&\text{(by (7.24))}\\
&\quad=\lim_{p\to p_0}\sqrt{\frac{s}{t}\frac{|\nu_t|(p)}{|\nu_s|(p)}}\\
&\quad\leqslant\sqrt{\frac{s}{t}\exp(-\varepsilon(s-t))} &&\text{(by (7.25))}\\
&\quad<\sqrt{\frac{s}{t}}.
\end{aligned}$$

This completes the proof. □

8. Convergence of harmonic maps rays

In this section, we complete the proof of Theorem 1.1. To prove the theorem, it will suffice, by Theorem 1.5, to prove that the family of harmonic maps $f_t\colon X_t\to Y$ converges in the sense of §4.2, as X_t degenerates along the harmonic map dual ray $\mathbf{hr}_{Y,\lambda}$. To this end, by Lemma 4.6, it suffices to show that every convergent sequence of the family

$$\{f_t\colon X_t\to Y\}_{t\geqslant 0}$$

shares the same limit. Let $f_{t_n}\colon X_{t_n}\to Y$ be an arbitrary convergent sequence with limit harmonic diffeomorphism $f_\infty\colon X_\infty\to Y\setminus\lambda_\infty$ for some geodesic lamination λ_∞.

8.1. Identifying limits of harmonic map rays with piecewise harmonic stretch lines

By Theorem 6.4, any (divergent) sequence of harmonic map rays through a fixed point subconverges to a Thurston geodesic ray/line. The goal of this subsection is to identify that limit ray with some piecewise harmonic stretch ray/line. Let $Y\in\mathcal{T}(S)$ be a fixed hyperbolic surface. Let $X_n\in\mathcal{T}(S)$ be a divergent sequence of Riemann surfaces such that the harmonic map rays $\mathbf{HR}_{X_n,Y}\colon[0,\infty)\to\mathcal{T}(S)$ converge to some Thurston geodesic $\mathbf{GR}\colon(0,\infty)\to\mathcal{T}(S)$. Note that our notation for the harmonic map rays sets

$$\mathbf{HR}_{X_n,Y}(1)\equiv Y.$$

Let $f_{n,t}\colon X_n\to\mathbf{HR}_{X_n,Y}(t)$ be the harmonic diffeomorphism in the given homotopy class.

Proposition 8.1. *With the notation and assumption as above, there exists a subsequence of X_n, still denoted by X_n for simplicity, with the following properties*:

(i) *The sequence $f_{n,1}\colon X_n\to Y$ converges to a harmonic diffeomorphism*

$$f_{\infty,1}\colon X_\infty\longrightarrow Y\setminus\lambda$$

for some punctured Riemann surface X_∞ and some chain recurrent geodesic lamination $\lambda\subset Y$.

(ii) *The limit geodesic $\mathbf{GR}\colon(0,\infty)\to\mathcal{T}(S)$ is the piecewise harmonic stretch line*

$$\mathrm{PSL}_{Y,\lambda,f_{\infty,1}}\colon(0,\infty)\longrightarrow\mathcal{T}(S)$$

defined by the triple $(Y,\lambda,f_{\infty,1})$.

(iii) *For any $s>0$, the sequence $f_{n,s}\colon X_n\to\mathbf{HR}_{X_n,Y}(s)$ converges to a harmonic diffeomorphism $f_{\infty,s}\colon X_\infty\to\mathbf{GR}(t)\setminus\lambda$ such that the Hopf differentials satisfy*

$$\mathrm{Hopf}(f_{\infty,s})=s\,\mathrm{Hopf}(f_{\infty,1}).$$

(iv) *For any $0<s<t$, the sequence $f_{n,t}\circ(f_{n,s})^{-1}\colon\mathbf{HR}_{X_n,Y}(s)\to\mathbf{HR}_{X_n,Y}(t)$ converges to a $\sqrt{t/s}$-Lipschitz homeomorphism $\mathbf{GR}(s)\to\mathbf{GR}(t)$ that extends*

$$f_{\infty,t}\circ(f_{\infty,s})^{-1}\colon\mathbf{GR}(s)\setminus\lambda\longrightarrow\mathbf{GR}(t)\setminus\lambda.$$

Proof. We begin with an outline of the argument. By Lemma 4.6, the sequence $f_{n,1}\colon X_n\to Y$ (sub)converges to a harmonic diffeomorphism $f_{\infty,1}\colon X_\infty\to Y\setminus\lambda$ for some punctured Riemann surface X_∞ and some chain recurrent geodesic lamination $\lambda\subset Y$. We inherit a complication from this construction: the surface X_∞ is a punctured surface and the limiting map $f_{\infty,1}$ of the sequence $f_{n,1}: X_n \to Y$ of maps, takes values in the open surface $Y \setminus \lambda$; in particular, the lamination λ is not in the image, so there is no a priori extension of the image $Y \setminus \lambda$ to the full surface Y.

We observe that the identical issue arises elsewhere along the Thurston geodesic $\mathbf{GR}$: for $Y_{n,s} = \mathbf{HR}_{X_n,Y}(s) \in \mathbf{HR}_{X_n,Y}$ along the ray $\mathbf{HR}_{X_n,Y}$, we have $\lim_{n\to\infty} Y_{n,s} = \mathbf{GR}_s$ (again because of the overall subconvergence of rays to a geodesic), but the limiting map $f_{n,s}: X_n \to Y_{n,s}$ has limit $f_{\infty,s}: X_\infty \to \mathbf{GR}_s \setminus \lambda$ (with limit Hopf differential $s\,\mathrm{Hopf}(f_\infty)$)—here we note again that the image is a subdomain $\mathbf{GR}_s \setminus \lambda$ of $\mathbf{GR}_s$.

Now, the Thurston geodesic $\mathbf{GR}_s$ has (non-canonical) corresponding homeomorphisms $\mathbf{L}_s: \mathbf{GR}_1 \to \mathbf{GR}_s$ which stretch exactly $\sqrt{s}$ along λ. (Here, we shall choose $\mathbf{L}_s$ as the limit of $f_{n,s} \circ (f_{n,1})^{-1}: Y \to Y_{n,s}$.) Separately, because $\mathbf{GR}_s$ is a compact surface, there is also an obvious extension $\mathbf{Ext}_s: \mathbf{GR}_s \setminus \lambda \to \mathbf{GR}_s$ (by continuity) of $\mathbf{GR}_s \setminus \lambda$ across λ to $\mathbf{GR}_s$. (When we carry out this plan, it will be convenient to realize this extension by taking limits along the images of the leaves of the vertical foliation of $\mathrm{Hopf}(f_{\infty,s})$. Equivalently, we will describe the extended foliation on $\mathbf{GR}_s$ of the pushforward of the vertical foliation of $\mathrm{Hopf}(f_{\infty,s})$ on $\mathbf{GR}_s \setminus \lambda$.)

Analogously, in terms of the harmonic maps, on the one hand, there is also a well-defined deformation of $f_{\infty,s}$ from f_∞ obtained by scaling the Hopf differentials, i.e. $\mathrm{Hopf}(f_{\infty,s}) = s\,\mathrm{Hopf}(f_\infty)$. In contrast to the case in the previous paragraph of the Thurston geodesic, there is however, not an immediately apparent extension of either $f_{\infty,s}$ or f_∞ across any punctures.

Thus, to prove (ii) in the proposition, evidently what we must show is that

$$\mathbf{L}_s(Y_1) = \mathbf{L}_s \circ \mathbf{Ext}_1 \circ f_{\infty,1}(X_\infty) = \mathbf{Ext}_s \circ \mathbf{L}_s \circ f_{\infty,1}(X_\infty). \tag{8.1}$$

Here we note that, in the right-hand side, we have that $f_{\infty,1}(X_\infty)$ has image $\mathbf{GR}_1 \setminus \lambda$, so that $\mathbf{L}_s$ is acting on the complement of the lamination λ, while on the left-hand side, we will have $\mathbf{L}_s$ acting on the extension of $f_{\infty,1}(X_\infty)$, and so acts as a homeomorphism of a compact surface. This commutation relation (8.1) allows us to identify the limit geodesic ray $\mathbf{GR}$ with the piecewise harmonic stretch ray defined by $(Y, \lambda, X_\infty, f_\infty)$ and complete the proof.

As a final remark, we note that we may expand the last displayed commutation relation between $\mathbf{L}$ and $\mathbf{Ext}$ and see it as an interchange of limits. In particular, the map $\mathbf{L}_s$ may be realized as the limit of the maps $f_{n,s} \circ [f_{n,1}]^{-1}: Y \to Y_{n,s}$. On the complement of the lamination λ, that composition limits on $f_{\infty,s} \circ [f_{\infty,1}]^{-1}: Y \setminus \lambda \to Y_s \setminus \lambda$. This then reduces what we must prove to a statement about the extension map $\mathbf{Ext}$.

This relates to the correspondence, across the puncture(s), between the ends of the vertical trajectories of $\mathrm{Hopf}(f_{\infty,s})$ (itself the limit of $\mathrm{Hopf}(f_{n,s})$): we must show that the extension map **Ext**, obtained as above by continuity across the lamination λ is also obtained as the (well-defined) limit of the image under $f_{\infty,s}$ of corresponding ends of the vertical trajectories we just described. [Put another way as an interchange of "temporal" and "spatial" limits of vertical leaves of Hopf differentials, because the vertical trajectories of $\mathrm{Hopf}(f_{n,s})$ realize the extension across λ for the image $f_{n,s}(X_n)$, we need to show that that limiting (in n) extension map across λ of the image under $f_{n,s}(X_n)$ of a vertical leaf of $\mathrm{Hopf}(f_{n,s})$ agrees with the (spatial) limit along the (disconnected) image of the end of that limiting corresponding vertical leaf of $\mathrm{Hopf}(f_{n,s})$.]

We now fill in the details of this sketch.

By Lemma 4.6, any sequence of maps $f_n: X_n \to Y$ contains a convergent subsequence, still denoted by the same notation for simplicity, which converges to a harmonic diffeomorphism $f_{\infty,1}: X_\infty \to Y \setminus \lambda$ for some punctured Riemann surface X_∞ and some chain recurrent geodesic lamination $\lambda \subset Y$. This proves the conclusion item (i).

We next address item (iii). By Theorem 1.5,[1] the harmonic diffeomorphism $f_{\infty,1}: X_\infty \to Y \setminus \lambda$ defines a piecewise harmonic stretch line $\mathrm{PSL}_{Y,\lambda,f_{\infty,1}}: (0,\infty) \to \mathcal{T}(S)$ such that the Hopf differential $\mathrm{Hopf}(f'_{\infty,s})$ of the induced harmonic diffeomorphism $f'_{\infty,s}: X_\infty \to \mathrm{PSL}_{Y,\lambda,f_{\infty,1}}(s) \setminus \lambda$ satisfies

$$\mathrm{Hopf}(f'_{\infty,s}) = s\,\mathrm{Hopf}(f_{\infty,1}). \tag{8.2}$$

Note that $\mathrm{PSL}_{Y,\lambda,f_{\infty,1}}(1) = Y$ and $f'_{\infty,1} = f_{\infty,1}$.

Now, by Lemma 7.7, we note the crucial point that the pushforward of the vertical foliation of $\mathrm{Hopf}(f'_{\infty,s})$ on $\mathrm{PSL}_{Y,\lambda,f_{\infty,1}}(s) \setminus \lambda$ extends to a unique measured foliation V_s on $\mathrm{PSL}_{Y,\lambda,f_{\infty,1}}(s)$. From the construction of the piecewise harmonic stretch lines (cf. the second paragraph in the proof of Theorem 1.5 in §7.5), we see that

$$V_s = \sqrt{s}\,V_1. \tag{8.3}$$

By assumption, the family of harmonic map rays $\mathbf{HR}_{X_n,Y}: [0,\infty) \to \mathcal{T}(S)$ locally uniformly converges to the Thurston geodesic ray $\mathbf{GR}: (0,\infty) \to \mathcal{T}(S)$. To simplify notations, we set $\mathbf{GR}_s = \mathbf{GR}(s)$. Note that $\mathbf{GR}_1 = Y$. For $s>0$, let $Y_{n,s} = \mathbf{HR}_{X_n,Y}(s) \in \mathbf{HR}_{X_n,Y}$ be the hyperbolic surface such that the Hopf differential of the harmonic diffeomorphism $f_{n,s}: X_n \to Y_{n,s}$ is $s\,\mathrm{Hopf}(f_{n,1})$. Then, $\mathbf{GR}_s = \lim_{n\to\infty} Y_{n,s}$. Furthermore, combining the fact that $f_{n,1}: X_n \to Y$ converges to $f_{\infty,1}: X_\infty \to Y \setminus \lambda$, the fact that

$$\mathrm{Hopf}(f_{n,s}) = s\,\mathrm{Hopf}(f_{n,1}) \quad \text{for all } s > 0,$$

[1] In this argument, we will use statements (a) and (b) of Theorem 1.5 that we proved in §7; we will not use the claim on analyticity in Theorem 1.5 relying on Lemma 7.4, whose proof we deferred until just after the present proof.

and the assumption that $\mathbf{HR}_{X_n,Y}\colon[0,\infty)\to\mathcal{T}(S)$ converges to $\mathbf{GR}\colon(0,\infty)\to\mathcal{T}(S)$ locally uniformly, we see that $f_{n,s}\colon X_n\to Y_{n,s}$ *subconverges* to a (not necessarily surjective) harmonic map $f_{\infty,s}\colon X_\infty\to\mathbf{GR}_s$ with $\mathrm{Hopf}(f_{\infty,s})=s\,\mathrm{Hopf}(f_{\infty,1})$. Note that by the second bullet property in Lemma 4.6, the geodesic lamination λ satisfies

$$\lambda=\lim_{R\to\infty}\lim_{n\to\infty}\overline{f_{n,1}(X_n\setminus\mathscr{P}_R(\mathrm{Hopf}(f_{n,1})))}, \tag{8.4}$$

where $\mathscr{P}_R(\mathrm{Hopf}(f_{n,1}))$ is the Minsky polygonal region of the Hopf differential $\mathrm{Hopf}(f_{n,1})$. Combining Theorem 3.9 and the fact (Lemma 6.1) that $f_{n,s}\circ(f_{n,1})^{-1}$ is a $\sqrt{s}$-Lipschitz homeomorphism homotopic to the identity, we see that

$$\lambda=\lim_{R\to\infty}\lim_{n\to\infty}\overline{f_{n,s}(X_n\setminus\mathscr{P}_R(\mathrm{Hopf}(f_{n,1})))}. \tag{8.5}$$

Applying a similar argument as in the proof of Lemma 4.6 shows that the image $f_{\infty,s}(X_\infty)$ is exactly $\mathbf{GR}_s\setminus\lambda$ and that $f_{\infty,s}$ is injective. In summary, the harmonic map $f_{\infty,s}\colon X_\infty\to\mathbf{GR}_s\setminus\lambda$ is a diffeomorphism with Hopf differential

$$\mathrm{Hopf}(f_{\infty,s})=s\,\mathrm{Hopf}(f_{\infty,1}). \tag{8.6}$$

Combined with (8.2) and Theorem 7.1 (with respect to the principal part of $s\Phi$ at the punctures of X_∞), this implies

$$\mathbf{GR}_s\setminus\lambda=\mathrm{PSL}_{Y,\lambda,f_{\infty,1}}(s)\setminus\lambda \quad\text{and}\quad f_{\infty,s}=f'_{\infty,s}.$$

In particular, for any fixed s, the map $f_{\infty,s}$ obtained as a limit of a subsequence in n of $f_{n,s}$ is independent of the choice of a convergent subsequence (in n) of $f_{n,s}\colon X_n\to Y_{n,s}$. Hence, the sequence $f_{n,s}\colon X_n\to Y_{n,s}$ converges to $f_{\infty,s}\colon X_\infty\to\mathbf{GR}_s\setminus\lambda$. This proves the conclusion item (iii) of the proposition.

To finish the proof of the identification $\mathbf{GR}_s=\mathrm{PSL}_{Y,\lambda,f_{\infty,1}}(s)$, it remains to show that $\mathbf{GR}_s\setminus\lambda$ extends to $\mathbf{GR}_s$ in the same way as $\mathrm{PSL}_{Y,\lambda,f_{\infty,1}}(s)\setminus\lambda$ extends to $\mathrm{PSL}_{Y,\lambda,f_{\infty,1}}(s)$. In other words, we need to show that the extended foliation $\mathbf{V}_s$ on $\mathbf{GR}_s$ of the pushforward of the vertical foliation of $\mathrm{Hopf}(f_{\infty,s})$ on $\mathbf{GR}_s\setminus\lambda$ is identical to extended foliation V_s on $\mathrm{PSL}_{Y,\lambda,f_{\infty,1}}(s)$ of the pushforward of the vertical foliation of $\mathrm{Hopf}(f'_{\infty,s})$ on $\mathrm{PSL}_{Y,\lambda,f_{\infty,1}}\setminus\lambda$. Recall that by (8.3), we have $V_s=\sqrt{s}V_1$. Note further that

$$\mathbf{GR}_1=Y=\mathrm{PSL}_{Y,\lambda,f_{\infty,1}}(1),$$

so $\mathbf{V}_1=V_1$. In view of these two identities, to prove $\mathbf{V}_s=V_s$, it suffices to show that

$$\mathbf{V}_s=\sqrt{s}\,\mathbf{V}_1.$$

For $s \geqslant 1$, from the construction of harmonic map rays, we see that the composition map $f_{n,s} \circ (f_{n,1})^{-1}: Y \to Y_{n,s}$, which is a $\sqrt{s}$-Lipschitz homeomorphism, sends leaves on Y of the pushforward of the vertical foliation of the Hopf differential $\mathrm{Hopf}(f_{n,1})$ by $f_{n,1}$ to corresponding leaves on $Y_{n,s}$ of the pushforward of the vertical foliation of the Hopf differential $\mathrm{Hopf}(f_{n,s}) = s\,\mathrm{Hopf}(f_{n,1})$ by $f_{n,s}$. Recall that

$$\lim_{n\to\infty} Y_{n,s} = \mathbf{GR}_s.$$

On the one hand, being $\sqrt{s}$-Lipschitz uniformly implies that $f_{n,s} \circ (f_{n,1})^{-1}: Y \to Y_{n,s}$ *subconverges* to a $\sqrt{s}$-Lipschitz map from Y to $\mathbf{GR}_s$. On the other hand, by the conclusion item (iii) proved above, for any $s \geqslant 1$, we have $f_{n,s}$ converges to $f_{\infty,s}$ as $n \to \infty$. Hence, the restriction $f_{n,s} \circ (f_{n,1})^{-1}|_{Y\setminus\lambda}$ converges to $f_{\infty,s} \circ (f_{\infty,1})^{-1}: Y \setminus \lambda \to \mathbf{GR}_s \setminus \lambda$. Furthermore, by Theorem 6.4, any limit of $f_{n,s} \circ (f_{n,1})^{-1}: Y \to Y_{n,s}$ stretches λ by the factor $\sqrt{s}$. Therefore, the limit map of any convergent subsequence of the sequence $f_{n,s} \circ (f_{n,1})^{-1}: Y \to Y_{n,s}$ is the $\sqrt{s}$-Lipschitz homeomorphism $\mathbf{L}_s: Y \to \mathbf{GR}_s$ that extends the limit composition map $f_{\infty,s} \circ (f_{\infty,1})^{-1}: Y \setminus \lambda \to \mathbf{GR}_s \setminus \lambda$, proving that $f_{n,s} \circ (f_{n,1})^{-1}$ converges to $\mathbf{L}_s$ and establishing the conclusion item (iv). Now, we also conclude from these considerations that the limit homeomorphism $\mathbf{L}_s: Y \to \mathbf{GR}_s$ sends leaves on Y of the extended foliation $\mathbf{V}_1$ (of the pushforward of the vertical foliation of the Hopf differential $\mathrm{Hopf}(f_{\infty,1})$ by $f_{\infty,1}$) to corresponding leaves on $\mathbf{GR}_s$ of the extended foliation $\mathbf{V}_s$ (of the pushforward of the vertical foliation of the Hopf differential $\mathrm{Hopf}(f_{\infty,s}) = s\,\mathrm{Hopf}(f_{\infty,1})$ by $f_{\infty,s}$). (In other words, this establishes the commutation relation (8.1).) This implies that

$$\mathbf{V}_s = \sqrt{s}\,\mathbf{V}_1, \tag{8.7}$$

where the factor $\sqrt{s}$ is due to the identity

$$\mathrm{Hopf}(f_{\infty,s}) = s\,\mathrm{Hopf}(f_{\infty,1}).$$

This completes the proof that $\mathbf{GR}_s = \mathrm{PSL}_{Y,\lambda,f_{\infty,1}}(s)$ for $s \geqslant 1$. The proof for $0 < s < 1$ is similar. Therefore, $\mathbf{GR}_s = \mathrm{PSL}_{Y,\lambda,f_{\infty,1}}(s)$ for $s > 0$, establishing the conclusion item (ii). The proof is now complete. □

Proof of Lemma 7.4. Let η be the horizontal foliation of Φ. Note that the pushforward $f_{t*}(t\eta)$ of $t\eta$ via $f_t: X \setminus D \to Y_t$ is an admissible measured foliation on Y_t. (See Definition 5.2 of "admissible measured foliations".) Consider $t=1$. By Proposition A.5, there exist

(i) a closed hyperbolic surface W and a chain recurrent geodesic lamination λ on W such that $W \setminus \lambda$ is the union of two or four isometric copies of (the crowned surface) Y,

(ii) a sequence of Riemann surfaces $X_n \in \mathcal{T}(W)$ and a sequence of harmonic diffeomorphisms $h_n: X_n \to W$ homotopic to the identity which converges to a harmonic diffeomorphism $h_\infty: X_\infty \to W \setminus \lambda$ (in the sense of Definition 4.5) from some punctured Riemann surface X_∞ such that, on each component X'_∞ of X_∞, the horizontal measured foliation of the Hopf differential of the restriction to $h_\infty|_{X'_\infty}: X'_\infty \to Y$ is exactly η.

Combining the second property with Theorem 5.7, we see that

$$X'_\infty = X \setminus D \quad \text{and} \quad h_\infty|_{X'_\infty} = f_1. \tag{8.8}$$

Now consider the sequence of harmonic map rays $\mathbf{HR}_{X_n,W}: [1,\infty) \to \mathcal{T}(W)$. In particular, $\mathbf{HR}_{X_n,W}(1) = W$ for all n. By Theorem 6.4, there exists a convergent subsequence of $\mathbf{HR}_{X_n,W}$, still denoted by $\mathbf{HR}_{X_n,W}$ for simplicity, that locally uniformly converges to some Thurston geodesic. By Proposition 8.1, that limit geodesic is the piecewise harmonic stretch line $\mathrm{PSL}_{W,\lambda,h_\infty}: (0,\infty) \to \mathcal{T}(W)$ defined by the triple (W, λ, h_∞). Let $g_{n,t}: X_n \to \mathbf{HR}_{X_n,W}(t)$ be the harmonic diffeomorphism in the given homotopy class. Then, by Lemma 6.1, for each $t > s > 0$, the composition map

$$g_{n,t} \circ (g_{n,s})^{-1}: \mathbf{HR}_{X_n,W}(s) \longrightarrow \mathbf{HR}_{X_n,W}(t)$$

is a $\sqrt{t/s}$-Lipschitz homeomorphism.

By Lemma 3.3, for each n, the harmonic map ray $\mathbf{HR}_{X_n,W}: [1,\infty) \to \mathcal{T}(W)$ is real analytic. Therefore, as the (locally uniform) limit of $\mathbf{HR}_{X_n,W}$, the piecewise harmonic stretch line $\mathrm{PSL}_{W,\lambda,h_\infty}: (0,\infty) \to \mathcal{T}(W)$ is also real analytic in $t > 0$. Note that the proof of Lemma 3.3 proves that the pullback metric (see (3.12)) of the hyperbolic metric of $\mathbf{HR}_{X_n,W}(t)$ via the harmonic diffeomorphism $g_{n,t}: X_n \to \mathbf{HR}_{X_n,W}(t)$ is real analytic in $t > 0$. According to [23, Theorem 3.1] (see also [83, Proposition 3.3]), we see that the family of harmonic diffeomorphisms $g_{n,t}$ is also real analytic in $t > 0$. In particular, the composition map

$$g_{n,t} \circ (g_{n,s})^{-1}: \mathbf{HR}_{X_n,W}(s) \longrightarrow \mathbf{HR}_{X_n,W}(t)$$

is real analytic in $t \in (s,\infty)$. By Proposition 8.1 (iii), we see that the sequence of harmonic diffeomorphisms $g_{n,t}: X_n \to \mathbf{HR}_{X_n,W}(t)$ converges to a harmonic diffeomorphism

$$g_{\infty,t}: X_\infty \longrightarrow \mathrm{PSL}_{W,\lambda,h_\infty}(t) \setminus \lambda$$

such that the Hopf differentials satisfy

$$\mathrm{Hopf}(g_{\infty,t}) = t\,\mathrm{Hopf}(h_\infty). \tag{8.9}$$

By Proposition 8.1 (iv), the sequence of $\sqrt{t/s}$-Lipschitz homeomorphisms $g_{n,t} \circ (g_{n,s})^{-1}$ converges to a $\sqrt{t/s}$-Lipschitz homeomorphism

$$\mathbf{L}_{s,t}: \mathrm{PSL}_{W,\lambda,h_\infty}(s) \longrightarrow \mathrm{PSL}_{W,\lambda,h_\infty}(t)$$

that extends

$$g_{\infty,t}\circ(g_{\infty,s})^{-1}\colon \mathrm{PSL}_{W,\lambda,h_\infty}(s)\setminus\lambda \longrightarrow \mathrm{PSL}_{W,\lambda,h_\infty}(t)\setminus\lambda.$$

Combined with the fact that $\mathbf{HR}_{X_n,W}\colon(0,\infty)\to\mathcal{T}(S)$ locally uniformly converges to

$$\mathrm{PSL}_{W,\lambda,h_\infty}\colon(0,\infty)\longrightarrow\mathcal{T}(W),$$

this implies that the limit lipschitz homeomorphism

$$\mathbf{L}_{s,t}\colon \mathrm{PSL}_{W,\lambda,h_\infty}(s)\longrightarrow \mathrm{PSL}_{W,\lambda,h_\infty}(t)$$

is also real analytic in $t\in(s,\infty)$. In particular, the restriction map

$$g_{\infty,t}\circ(g_{\infty,s})^{-1}\colon \mathrm{PSL}_{W,\lambda,f_\infty}(s)\setminus\lambda \longrightarrow \mathrm{PSL}_{W,\lambda,f_\infty}(t)\setminus\lambda$$

is also real analytic in $t\in(s,\infty)$. Combining (8.8), (8.9) and Theorem 7.1, we see that $\mathrm{PSL}_{W,\lambda,f_\infty}(t)\setminus\lambda$ is the union of two or four isometric copies of Y_t, and that on each component X'_∞ $(=X\setminus D$ by (8.8)) of X_∞, we have $g_{\infty,t}|_{X'_\infty}=f_t$. In particular, on each component of $\mathrm{PSL}_{W,\lambda,h_\infty}(1)\setminus\lambda$, which is also rewritten as $W\setminus\lambda$, the restriction of the limit map $\mathbf{L}_{s,t}\colon\mathrm{PSL}_{W,\lambda,h_\infty}(s)\to\mathrm{PSL}_{W,\lambda,h_\infty}(t)$ coincides with the composition map

$$f_t\circ(f_s)^{-1}\colon Y_s\longrightarrow Y_t.$$

Therefore, the analyticity of $\mathbf{L}_{s,t}$ and the ray $\mathrm{PSL}_{W,\lambda,f_\infty}\colon(0,\infty)\to\mathcal{T}(W)$ implies that the surface Y_t and the map $f_t\circ(f_s)^{-1}\colon Y_s\to Y_t$ along with its Lipschitz homeomorphic extension from the closure of Y_s to the closure of Y_t (either obtained from Lemma 7.3 or induced from $\mathbf{L}_{s,t}$) are real analytic in $t\in(s,\infty)$. The analyticity of f_t follows by precomposing $f_t\circ(f_s)^{-1}$ by f_s. This completes the proof. ☐

The proof of Theorem 1.5 is now complete.

8.2. Identifying the limit horizontal foliation

We now turn to the proof of Theorem 1.1. To begin, let $X_{t_n}\in\mathbf{hr}_{Y,\lambda}$ and $f_{t_n}\colon X_{t_n}\to Y$ be an arbitrary convergent sequence with limit harmonic diffeomorphism $f_\infty\colon X_\infty\to Y\setminus\lambda_\infty$ for some geodesic lamination λ_∞ (Lemma 4.6).

LEMMA 8.2. *With the notation introduced as above, we have* $\lambda_\infty=\lambda$.

Proof. Let $\mathscr{P}_{R_n}(\Phi_{t_n})$ be the Minsky's polygonal region on X_{t_n} with $t_n\to\infty$ and $R_n:=t_n^{1/4}\to\infty$ as $n\to\infty$. Then, by Theorem 3.9, the complement $f_{t_n}(X_{t_n}\setminus\mathscr{P}_{R_n}(\Phi_{t_n}))$ is contained in an ε_n neighbourhood of λ on Y, where $\varepsilon_n\to0$ as $n\to\infty$. Hence, $\lambda_\infty\subset\lambda$

and $f_\infty(X_\infty)\supset Y\setminus\lambda$. For the other direction, consider the minimal components of λ, denoted by $\lambda^i, ..., \lambda^k$. Note that the horizontal foliation of Φ_{t_n} is $t_n\lambda$. Let $X^1_{t_n}, ..., X^k_{t_n}$ be the complementary components of the critical graph (cf. Definition 2.7) of the horizontal foliation of Φ_{t_n}, labelled in such a way that the restriction to $X^i_{t_n}$ of the horizontal foliation of Φ_{t_n} corresponds to the minimal component $t_n\lambda^i$ of $t_n\lambda$. By (3.23),

$$2\|\Phi_{t_n}|_{X^i_{t_n}}\|\geqslant \ell_Y(t_n\lambda^i)-C$$

for some constant C. Recall that $R_n=t_n^{1/4}$. By item (v) of Theorem 3.7, the area of $\mathscr{P}_{R_n}(\Phi_{t_n})$ is at most $c\sqrt{t_n}$, which is much less than $\|\Phi_{t_n}|_{X^i_{t_n}}\|$. Hence, the subsurface $X^i_{t_n}$ is not contained in $\mathscr{P}_{R_{t_n}}(\Phi_{t_n})$. In particular, from item (vi) of Theorem 3.7, we see that there exists a horizontal leaf segment $l^i_{t_n}$ of $\Phi_{t_n}|_{X^i_{t_n}}$ in the boundary of $\mathscr{P}_{R_{t_n}}(\Phi_{t_n})$ with length $|l^i_{t_n}|\geqslant K_3R_n=K_3t_n^{1/4}$ for some constant K_3. By Theorem 3.9, the image $f_{t_n}(l^i_{t_n})$ is nearly a geodesic segment contained in the ε_n neighbourhood of the minimal component $t_n\lambda^i$. That λ^i is minimal and that $t_n\to\infty$ imply that $f_{t_n}(l^i_{t_n})$ converges to the whole component λ^i as $n\to\infty$. Therefore, the limit lamination λ_∞, which is the limit of $f_{t_n}(X_{t_n}-\mathscr{P}_{R_{t_n}}(\Phi_{t_n}))$ as $n\to\infty$, contains λ. Combined with the discussion in the beginning of the proof, this proves that $\lambda_\infty=\lambda$. □

A direct consequence of the above lemma is that X_∞ and $Y\setminus\lambda$ are homeomorphic. Let $Y^1, ..., Y^m$ be the components of $Y\setminus\lambda$. Let $X_\infty=X^1\cup X^2\cup...\cup X^m$ with X^i homeomorphic to Y^i. Set $f^i:=f_\infty|_{X^i}$.

Lemma 8.3. *The critical graph* (*Definition* 2.7) *of the Hopf differential* $\mathrm{Hopf}(f^i)$ *is connected. Moreover, the complement of the critical graph of* $\mathrm{Hopf}(f^i)$ *consists of half-infinite cylinders corresponding to the closed geodesic boundary components of* Y^i *and half-planes corresponding to the ideal geodesic boundary components of* Y^i.

Proof. Let λ^i be the horizontal measured foliation of $\mathrm{Hopf}(f^i)$.

Claim 1. *If* λ *contains a simple closed curve* α, *then the corresponding* (*maximal*) *cylinder component* A_{t_n} *of the horizontal foliation of* $\mathrm{Hopf}(f_{t_n})$ *limits on two half-infinite cylinders on* $\mathrm{Hopf}(f_\infty)$.

Proof. As $n\to\infty$, the height of A_{t_n} on X_{t_n} goes to infinity. Combining with the Minsky's estimate (Theorem 3.9), we see that the circumference of A_{t_n} converges to half the hyperbolic length of the geodesic representative of α on Y, which is both finite and positive. Therefore, the cylinder A_{t_n} converges to two half-infinite cylinders on X_∞ with circumferences equal to half the hyperbolic length of the geodesic representative of α on Y. This proves Claim 1. □

Claim 2. *For each i, the measured foliation λ^i has no compact components* (*see* §2.8 *for the definition*).

Proof. Otherwise, suppose that λ^i contains a compact component λ^i_0. Since the horizontal foliation of $\mathrm{Hopf}(f_t)$ is simply $t\lambda$, we see that λ^i_0 has to be a component of $t\lambda$. Hence, the scaling λ^i_0/t is a component of λ, which goes to zero as t goes to infinity. As a consequence, λ^i_0 is not a component of λ, which contradicts that λ^i_0 is a component of $t\lambda$. This proves Claim 2. □

By Claim 2, we see that each component of the complement of the critical graph of $\mathrm{Hopf}(f^i)$ in X^i is either a half-infinite cylinder, a half-plane, or an infinite strip.

Claim 3. *The complement of the critical graph of* $\mathrm{Hopf}(f^i)$ *contains no infinite-strip components* (*see* §2.8 *for the definition*).

Proof. Suppose to the contrary that the complement of the critical graph of $\mathrm{Hopf}(f^i)$ contains a component C which is an infinite strip. Let β be a saddle connection in C which connects two zeros z^+, z^- of $\mathrm{Hopf}(f^i)$ belonging to different boundary components of C. Let U be a δ-neighbourhood of β on X^i under the flat metric $|\mathrm{Hopf}(f^i)|$ which contains no zeros of $\mathrm{Hopf}(f^i)$ other than $z^\pm$. As the flat surface $(X_{t_n}, \mathrm{Hopf}(f_{t_n}))$ converges to $(X_\infty, \mathrm{Hopf}(f_\infty))$, there exists a sequence of approximating maps $\eta_{t_n}\colon U \to X_{t_n}$ homeomorphic to the corresponding images such that the following properties hold:

- $\eta_{t_n}(z^\pm)$ are zeros of $\mathrm{Hopf}(f_{t_n})$;
- the pullback metrics

$$\eta^*_{t_n}(|\mathrm{Hopf}(f_{t_n})|$$

converge to the restriction of $|\mathrm{Hopf}(f^i)|$ to U.

(Here, note that there may be many choices for $\eta_{t_n}(z^\pm)$, say if multiple zeros of $\mathrm{Hopf}(f_{t_n})$ converge to $z^\pm$: the precise choice will not be important in what follows.) The assumption that U contains no zeros of $\mathrm{Hopf}(f^i)$ other than $z^\pm$ implies that, as n goes to infinity, every zero contained in the image $\eta_{t_n}(U)$ would collapse to z^+ or z^-. Since β is a geodesic segment which contains no zeros of $\mathrm{Hopf}(f_\infty)$ in the interior, we may modify η_{t_n} in such a way that the geodesic representative of $\eta_{t_n}(\beta)$ is also a saddle connection for large enough n. Let us fix such a large enough $\bar{\mathbf{n}}$. It then follows from the recurrence property of leaves of measured foliations on closed surfaces that there exists a closed curve γ which is a concatenation of a horizontal leaf segment $\zeta_{t_{\bar{\mathbf{n}}}}$ and a subset segment $\xi_{t_{\bar{\mathbf{n}}}}$ of $\eta_{t_{\bar{\mathbf{n}}}}(\beta)$ centered near the midpoint of $\eta_{t_{\bar{\mathbf{n}}}}(\beta)$ and with length $0<|\xi_{t_{\bar{\mathbf{n}}}}|<\frac{1}{2}|\beta|$ (that $0<|\xi_{t_{\bar{\mathbf{n}}}}|$ follows from Claim 1). In particular, the intersection number of γ and the horizontal measured foliation of $\mathrm{Hopf}(f_{t_{\bar{\mathbf{n}}}})$, which is equivalent to $t_{\bar{\mathbf{n}}}\lambda$, satisfies

$$0 < i(\gamma, t_{\bar{\mathbf{n}}}\lambda) < |\xi_{t_{\bar{\mathbf{n}}}}| < \frac{|\beta|}{2}.$$

Therefore,

$$i(\gamma, t_n\lambda) = \frac{t_n}{t_{\bar{\mathbf{n}}}} i(\gamma, t_{\bar{\mathbf{n}}}\lambda) \to \infty, \quad \text{as } n \to \infty, \quad \text{because } \frac{t_n}{t_{\bar{\mathbf{n}}}} \to \infty. \tag{8.10}$$

Notice that, for each $n > \bar{\mathbf{n}}$ the (topological) horizontal foliation of $\mathrm{Hopf}(f_{t_n})$ can be obtained from that of $\mathrm{Hopf}(f_{t_{\bar{\mathbf{n}}}})$ by a homeomorphism followed by a sequence of Whitehead moves. This implies that γ can also be realized on X_{t_n} as a concatenation of a horizontal leaf segment ζ_{t_n} and a subsegment ξ_{t_n} of $\eta_{t_n}(\beta)$, which gives

$$i(\gamma, t_n\lambda) < |\eta_{t_n}(\beta)| \to |\beta| \quad \text{as } n \to \infty.$$

This contradicts (8.10), completing the proof of Claim 3. □

The lemma then follows from Claims 1–3. □

As a direct consequence of Lemma 8.3, we have the following result.

LEMMA 8.4. *The dual tree of the lift of the horizontal measured foliation of the Hopf differential* $\mathrm{Hopf}(f^i)$ *to the universal cover consists of exactly one vertex and countably many half-infinite edges corresponding to the boundary of* $\widetilde{Y}^i$.

Consider the family of harmonic maps $f_t: X_t \to Y$ with Hopf differential

$$\mathrm{Hopf}(f_t) = \Phi_t.$$

The image of $X_t \setminus \mathscr{P}_R(\Phi_t)$ under the map f_t is a (thickened) train track we denote by $\tau_{t,R}$. An ε *train track* that carries λ is a train track that carries λ and is contained in the ε-neighbourhood of λ. We now combine Lemma 8.3 and Theorem 3.9.

LEMMA 8.5. (Train-track approximation) *For any* $\varepsilon > 0$, *there exists* $T_0 > 0$ *such that, for any* $t > T_0$, *the train track* $\tau_{t,t^{1/4}}$ *in* Y *corresponding to* $X_t \setminus \mathscr{P}_{t^{1/4}}(\Phi_t)$ *is an* ε *train track which carries* λ.

Proof. For any $\varepsilon > 0$, by Theorem 3.9, there exists $T_0 > 0$ such that, for any $t > T_0$, the thickened train track $\tau_{t,t^{1/4}}$, being the image of $X_t \setminus \mathscr{P}_{t^{1/4}}(\Phi_t)$ under the harmonic diffeomorphism $f_t: X_t \to Y$, is contained in the ε-neighbourhood of λ. It remains to show that $\tau_{t,t^{1/4}}$ carries λ.

By Lemma 8.3, there exists $T_0 > 0$ such that every non-critical horizontal leaf in $\mathscr{P}_{t^{1/4}}(\Phi_t)$ is either

- a closed leaf homotopic to one of the boundary components of $\mathscr{P}_{t^{1/4}}(\Phi_t)$, or
- contained in a maximal leaf which is homotopic *rel* $\partial_v \mathscr{P}_{t^{1/4}}(\Phi_t)$ to a horizontal boundary leaf of $\mathscr{P}_{t^{1/4}}(\Phi_t)$, where $\partial_v \mathscr{P}_{t^{1/4}}(\Phi_t)$ comprises the vertical boundary segments of $\partial \mathscr{P}_{t^{1/4}}(\Phi_t)$, or
- contained in a horizontal strip of finite height.

For the third case, since all these strips are strips of $t\lambda$, the topological type of these strips stabilizes as $t\to\infty$. Let Strip_t be such a strip. By Lemma 3.8, each maximal leaf in Strip_t has length at least $K_3'R_t=K_3't^{1/4}$. By Theorem 3.7 (v), it follows that Strip_t has height $d_t\leqslant C't^{1/4}$. Let β_t be a saddle connection in Strip_t connecting the two distinct boundary components of Strip_t. Then, by the recurrence property of horizontal leaves of Φ_t, there exists a closed curve γ which is a concatenation of a subsegment β_t' of β_t and a leaf segment ζ_t of $t\lambda$ which is the horizontal foliation of Φ_t. In particular, we have $0<i(\gamma,t\lambda)<d_t$. Equivalently, $0<i(\gamma,\lambda)<d_t/t$. Letting $t\to\infty$ yields $0<i(\gamma,\lambda)=0$, which is impossible. So the third case cannot happen. For the first two cases, the image of every non-critical horizontal leaf of $\mathscr{P}_{t^{1/4}}(\Phi_t)$ is carried by the train-track $\tau_{r,t^{1/4}}$ corresponding to $X_t\setminus\mathscr{P}_{t^{1/4}}(\Phi_t)$. Therefore, the train track $\tau_{t,t^{1/4}}$ carries λ for all $t>T_0$. □

Remark 8.6. Note that, for Hopf differentials and choices of size R of Minsky regions in less restrictive settings than we are considering here, the train track τ in Y corresponding to the complement of $\mathscr{P}_R$ is not sufficient to carry λ, because it may not carry those bi-infinite leaves which are contained in $\mathscr{P}_R$.

We will use this lemma in §11.

8.3. Proof of Theorem 1.1

Proof of Theorem 1.1. The proof will be divided into two steps. In the first step, we will establish the convergence of harmonic map rays $\mathbf{HR}_{X_t,Y}$ when X_t diverges along harmonic map dual rays. In the second step, we will identify the limit geodesic with the Thurston stretch line provided that the lamination λ is maximal.

Step 1. Convergence when X_t diverges along the harmonic map dual ray $\mathbf{hr}_{Y,\lambda}$.

Let $X_t=\mathbf{hr}_{Y,\lambda}(t)$. Consider the family of harmonic maps $f_t\colon X_t\to Y$. By Lemma 4.6, any sequence of maps $f_{t_n}\colon X_{t_n}\to Y$ contains a convergent subsequence. For each component Y^i of $Y\setminus\lambda$, by Lemma 8.4, the dual tree of the lift on $\widetilde{Y}^i$ of the pushforward of the horizontal foliation of any limit Hopf differential is the regular tree with one vertex and countably many half-infinite edges corresponding to the boundary edges of $\widetilde{Y}^i$. By Theorem 5.7, the limit harmonic maps of all convergent sequences of $f_t\colon X_t\to Y$ are the same, say $f_\infty\colon X_\infty\to Y\setminus\lambda$. By Theorem 6.4, any sequence of the family of harmonic map rays $\mathbf{HR}_{X_t,Y}\colon[1,\infty)\to\mathcal{T}(S)$ contains a subsequence which locally uniformly converges to some Thurston geodesic ray. However, by Proposition 8.1, that limit Thurston geodesic ray of any convergent subsequence of $\mathbf{HR}_{X_t,Y}\colon[1,\infty)\to\mathcal{T}(S)$ is the piecewise harmonic stretch ray $\mathbf{PSR}_{Y,\lambda,f_\infty}\colon[1,\infty)\to\mathcal{T}(S)$ defined by (Y,λ,f_∞). (Note that $\mathbf{HR}_{X_t,Y}\colon[1,\infty)\to\mathcal{T}(S)$

is the subray of $\mathbf{HR}_{X_t,Y}\colon[0,\infty)\to\mathcal{T}(S)$ starting from Y and that $\mathbf{PSR}_{Y,\lambda,f_\infty}$ is the subray of $\mathrm{PSL}_{Y,\lambda,f_\infty}$ starting from Y.) Hence, the family of harmonic map rays $\mathbf{HR}_{X_t,Y}$ converges to the piecewise harmonic stretch ray $\mathbf{PSR}_{Y,\lambda,f_\infty}$ (independent of the choice of subsequences). That the convergence is locally uniform follows from Theorem 1.3.

Step 2. Identifying the limiting geodesic with the Thurston stretch line for maximal geodesic laminations.

This follows from Lemma 7.10 and completes the proof. □

For convenience of a later reference, we summarize the following result from the discussion in step 1 of the proof of Theorem 1.1. Let $Y\in\mathcal{T}(S)$ be a hyperbolic surface and let λ be a measured geodesic lamination on Y. Let η be the admissible measured foliation (Definition 5.2) on $Y\setminus\lambda$ that comprises half-infinite cylinders foliated by closed leaves parallel to closed geodesic boundary components of $Y\setminus\lambda$ and half-planes foliated by bi-infinite leaves parallel to ideal boundary geodesics of $Y\setminus\lambda$.

THEOREM 8.7. *Let Y, λ and η be as above. Then, there exists a unique* (*possibly disconnected*) *punctured Riemann surface X_∞, homeomorphic to $Y\setminus\lambda$, and a unique harmonic diffeomorphism $f_\infty\colon X_\infty\to Y\setminus\lambda$, homotopic to the identity map, such that the pushforward to $Y\setminus\lambda$ of the horizontal foliation of the Hopf differential of f_∞ is measure equivalent to η.*

We close this section with a discussion when X_t diverges along a Teichmüller geodesic.

THEOREM 8.8. *Let $Y\in\mathcal{T}(S)$ and let λ be a maximal measured geodesic lamination on Y. Let $X_t\in\mathbf{TR}_{Y,\lambda}$. Then, the path of harmonic map rays $\mathbf{HR}_{X_t,Y}$ converges to the Thurston stretch line $\mathbf{SL}_{Y,\lambda}$.*

Proof. By Theorem 6.4, for any divergent sequence $X_{t_n}\in\mathbf{TR}_{Y,\lambda}$, the sequence

$$\mathbf{HR}_{X_{t_n},Y}\colon[0,\infty)\longrightarrow\mathcal{T}(S)$$

of harmonic map rays contains a subsequence that converges to a Thurston geodesic. By Proposition 8.1, that limit Thurston geodesic is a piecewise harmonic stretch line $\mathrm{PSL}_{Y,\lambda',f_\infty}\colon(0,\infty)\to\mathcal{T}(S)$ defined by a chain recurrent geodesic lamination λ' and a harmonic diffeomorphism $f_\infty\colon X_\infty\to Y\setminus\lambda'$ from some (possibly disconnected) punctured Riemann surface X_∞. For simplicity, we assume that $\mathbf{HR}_{X_{t_n},Y}$ itself converges. By Theorem 1.5, the piecewise harmonic stretch line $\mathrm{PSL}_{Y,\lambda',f_\infty}$ maximally stretches exactly λ'. By [61, Theorem 8.1], we see that, up to considering a subsequence if necessary, the horizontal measured foliation of the harmonic map $h_{t_n}\colon X_{t_n}\to Y$ homotopic to the identity projectively converges to a measured foliation supported on λ. Then, Theorem 6.4

implies $\lambda \subset \lambda'$. Since λ is maximal, it follows that $\lambda' = \lambda$. By Lemma 7.10, we see that the limit geodesic $\mathrm{PSL}_{Y,\lambda',f_\infty}$ is the Thurston stretch line $\mathbf{SR}_{Y,\lambda}$. The theorem then follows from the arbitrariness of the convergence subsequence of $\mathbf{HR}_{X_t,Y}$. □

9. Convergence to Teichmüller rays

Let $X \in \mathcal{T}(S)$ be a fixed Riemann surface and Φ be a holomorphic quadratic differential on X. Let $\mathbf{HR}_{X,\Phi}$ be the harmonic map ray defined by X and Φ. For $s \geqslant 0$, let $Y_s = \mathbf{HR}_{X,\Phi}(s) \in \mathbf{HR}_{X,\Phi}$ be the hyperbolic surface such that the Hopf differential of the harmonic map from X to Y_s is $s\Phi$. Let T_h be the $\mathbb{R}$-tree dual to the horizontal measured foliation of $\tilde{\Phi}$, the lift of Φ to the universal cover $\widetilde{X}$. Then, $\widetilde{X}$ is the minimal surface in $\widetilde{Y}_s \times (T_h, 2s^{1/2}d)$, also a rescaled minimal surface in $s^{-1/2}\widetilde{Y}_s \times (T_h, 2d)$. As $s \to \infty$, $\widetilde{X}$ is exactly the minimal surface in $T_v \times T_h$, where T_v is the $\mathbb{R}$-tree dual to the vertical measured foliation of $\tilde{\Phi}$. Let $X_{s,t} \in \mathbf{hr}_{Y_s,\sqrt{s}\lambda}$ be such that $\mathrm{Hor}(\mathrm{Hopf}(X_{s,t} \to Y_s) = t\sqrt{s}\lambda$. Then, $X_{s,1} \equiv X$ for all $s > 0$. The goal of this section is to prove the following.

THEOREM 9.1. *For any fixed X and λ, as $s \to \infty$, the family of harmonic map dual rays*

$$\mathbf{hr}_{Y_s,\sqrt{s}\lambda} \colon [1,\infty) \longrightarrow \mathcal{T}(S)$$

locally uniformly converge to the Teichmüller geodesic ray

$$\mathbf{TR}_{X,\Phi} \colon [1,\infty) \longrightarrow \mathcal{T}(S)$$

with $\mathrm{Hor}(\Phi) = \lambda$.

Convention. In the remainder of this section, to simplify the notation, we will denote the dual tree $(T_\eta, 2d)$ by T_η.

9.1. Minimal surfaces in the product of trees

Let α and β be a pair of transverse measured foliations on the closed surface S, that is, for any homotopically non-trivial simple closed curve γ we have

$$i(\alpha,\gamma) + i(\beta,\gamma) > 0.$$

By [28, Theorem 5.1], there exists a unique holomorphic quadratic differential Ψ whose horizontal and vertical measured foliations are α and β, respectively. Let T_α and T_β be the dual trees of α and β, respectively. Define the energy map $E(\bullet, T_\alpha \times T_\beta) \colon \mathcal{T}(S) \to \mathbb{R}$ which associates to $Z \in \mathcal{T}(S)$ the (equivariant) energy of the equivariant harmonic map $\widetilde{Z} \to T_\alpha \times T_\beta$: the integral of the energy density over a fundamental domain of $\widetilde{Z}$ under the Fuchsian group defining Z (see [99, p. 111]).

LEMMA 9.2. *The function $E(\bullet, T_\alpha \times T_\beta)\colon \mathcal{T}(S) \to \mathbb{R}$ is proper. Moreover,*

$$E(Z, T_\alpha \times T_\beta) \geqslant 4\|\Psi\|,$$

where the equality holds if and only if Z is the underlying Riemann surface of the quadratic differential Ψ.

Proof. Recall that

$$E(Z, T_\alpha \times T_\beta) = E(Z, T_\alpha) + E(Z, T_\beta) = 2\,\mathrm{Ext}_Z(\alpha) + 2\,\mathrm{Ext}_Z(\beta),$$

where the second equation follows from (3.4). The properness then follows from the fact that $\mathrm{Ext}_Z(\alpha)+\mathrm{Ext}_Z(\beta)$ is a proper function over $\mathcal{T}(S)$. Moreover,

$$\begin{aligned} \mathrm{Ext}_Z(\alpha)+\mathrm{Ext}_Z(\beta) &\geqslant 2\sqrt{\mathrm{Ext}_Z(\alpha)\cdot \mathrm{Ext}_Z(\beta)} \\ &\geqslant 2i(\alpha,\beta) && \text{(by [28, Theorem 5.1])} \\ &= 2\|\Psi\|, \end{aligned}$$

where the equalities hold if and only if

- $\mathrm{Ext}_Z(\alpha)=\mathrm{Ext}_Z(\beta)$,
- Z is the underlying Riemann surface of Ψ. □

9.2. Estimating the energy of harmonic maps to trees

For any $Z \in \mathcal{T}(S)$ and each measured foliation μ on Z, we denote by $E(Z, T_\mu)$ the equivariant energy of the harmonic map from $\widetilde{Z}$ to the dual tree of the lift of μ. By (3.4), we have $E(Z, T_\mu) = 2\,\mathrm{Ext}_Z(\mu)$.

LEMMA 9.3. *Let $Y_s \in \mathbf{HR}_{X,\Phi}$. Let $\bar\lambda$ be the vertical measured lamination of Φ. Then, for any $Z \in \mathcal{T}(S)$, we have $E(Z, T_{\bar\lambda}) \leqslant E(Z, s^{-1}Y_s)$, where $E(Z, s^{-1}Y_s)$ is the energy of the harmonic diffeomorphism from Z to $s^{-1}Y_s$ (see §3.1).*

Proof. We define a family of equivariant projection maps

$$j_s\colon s^{-1}\widetilde{Y}_s \longrightarrow T_{\bar\lambda}$$

as follows. Recall that, on the natural coordinates of Φ on X, the hyperbolic metric on Y_s can be written as

$$f_s^* Y_s = 2s(\cosh \mathscr{G}(z,s)+1)\,dx^2 + 2s(\cosh \mathscr{G}(z,s)-1)\,dy^2,$$

where $f_s\colon X\to Y_s$ is the unique harmonic map. Let $j_s\colon s^{-1}\widetilde{Y}_s\longrightarrow T_{\bar\lambda}$ be the projection map along the vertical leaves of $\tilde\Phi$, the lift of Φ to $\widetilde{X}$. Then, j_s collapses the vertical leaves while it scales the horizontal leaves by a factor of

$$\sqrt{\frac{2}{\cosh\mathscr{G}(z,s)+1}}<1$$

at $f_s(z)\in\widetilde{Y}_s$ (with respect to the hyperbolic metric on $\widetilde{Y}_s$).

Let $F_s\colon\widetilde{Z}\to s^{-1}\widetilde{Y}_s$ be the harmonic map, then

$$E(Z,T_{\bar\lambda})\leqslant E(j_s\circ F_s)\leqslant E(F_s)=E(Z,s^{-1}Y_s). \qquad \square$$

9.3. Proof of Theorem 9.1

Proof of Theorem 9.1. Let $\bar\lambda$ be the vertical measured lamination of Φ. Let $\Phi_{s,t}$ be the Hopf differential of $X_{s,t}\to Y_s$. Then, the Hopf differential of $\widetilde{X}_{s,t}\to T_{\sqrt{s}t\lambda}$ is $-\tilde\Phi_{s,t}$, the lift of $-\Phi_{s,t}$. Consider the equivariant harmonic map $\widetilde{X}_{s,t}\to T_{\bar\lambda}\times tT_\lambda$. By Lemma 9.3, we have

$$\begin{aligned}
&E(X_{s,t},T_{\bar\lambda}\times T_{t\lambda})\\
&\quad=E(X_{s,t},T_{\bar\lambda})+E(X_{s,t},T_{t\lambda})\\
&\quad\leqslant E(X_{s,t},s^{-1}Y_s)+E(X_{s,t},T_{t\lambda}) &&\text{(by Lemma 9.3)}\\
&\quad=s^{-1}E(X_{s,t},Y_s)+s^{-1}E(X_{s,t},T_{\sqrt{s}t\lambda})\\
&\quad=s^{-1}E(X_{s,t},Y_s)+2s^{-1}\|\Phi_{s,t}\|\\
&\quad\leqslant s^{-1}(\ell_{Y_s}(\sqrt{s}t\lambda)+C)+s^{-1}(\ell_{Y_s}(\sqrt{s}t\lambda)+C) &&\text{(by Lemma 3.12)}\\
&\quad=2s^{-1}(t\ell_{Y_s}(\sqrt{s}\lambda)+C)\\
&\quad\leqslant 2s^{-1}(2t\|s\Phi\|+tC)+C) &&\text{(by Lemma 3.12)}\\
&\quad=4t\|\Phi\|+2s^{-1}C(t+1). &&(9.1)
\end{aligned}$$

Combined with Lemma 9.2, this implies that, for any fixed t,

$$E(X_{s,t},T_{\bar\lambda}\times T_{t\lambda})\to 4t\|\Phi\|=4i(\bar\lambda,t\lambda),\quad \text{as } s\to\infty.$$

It then follows from Lemma 9.2 that $X_{s,t}\to\mathbf{TR}_\Phi(t)$, as $s\to\infty$.

To see the locally uniform convergence, consider the function $\psi\colon[0,\infty)\to[0,\infty)$ defined as follows. Let $X_{\infty,t}\in\mathbf{TR}_{X,\Phi}$ be the Riemann surface underlying the quadratic differential whose horizontal and vertical measured foliations are $t\lambda$ and $\bar{\lambda}$, respectively. Then, for any $Z\in\mathcal{T}(S)$, we have $E(Z,T_{\bar{\lambda}}\times T_{t\lambda})\geqslant E(X_{\infty,t},T_{\bar{\lambda}}\times T_{t\lambda})$. For any $t_0\geqslant 1$, let

$$\psi_{t_0}(r)=\max_{1\leqslant t\leqslant t_0}\max\{d_T(Z,X_{\infty,t}):E(Z,T_{\bar{\lambda}}\times T_{t\lambda})-E(X_{\infty,t},T_{\bar{\lambda}}\times T_{t\lambda})\leqslant r\}. \tag{9.2}$$

Then, ψ is continuous and increasing in r. By (9.1), we see that

$$E(X_{s,t},T_{\bar{\lambda}}\times T_{t\lambda})-E(X_{\infty,t},T_{\bar{\lambda}}\times T_{t\lambda})\leqslant 2s^{-1}C(t_0+1)\to 0, \quad \text{as } s\to\infty.$$

Then,

$$d_T(X_{s,t},X_{\infty,t})\leqslant\psi_{t_0}(2s^{-1}C(t_0+1))\to 0$$

uniformly in $t\in[1,t_0]$, as $s\to\infty$. □

10. Convergence to Teichmüller disks

In this section, we introduce two models of harmonic dual disks, both of which will converge to Teichmüller disks.

Let $Y_{s,\theta}$ be the hyperbolic surface such that

$$\mathrm{Hopf}(X,Y_{s,\theta})=se^{2i\theta}\Phi.$$

Let λ_θ be the horizontal foliation of $e^{2i\theta}\Phi$.

M1. Let

$$\mathbf{HD}_{X,\Phi,s}=\bigcup_{0\leqslant\theta\leqslant\pi}\mathbf{hr}_{Y_{s,\theta},\lambda_\theta}$$

denote the (variable target) harmonic map dual disk.

M2. Let

$$\widehat{\mathbf{HD}}_{X,\Phi,s}=\bigcup_{-\pi/2<\theta<\pi/2}\mathbf{hr}_{Y_s,\lambda_\theta}$$

denote the (fixed target) harmonic map dual disk.

In both versions, there is a dependence of the lamination λ_θ on θ. In the first version **M1**, the family of terminal points $Y_{s,\theta}$ also changes with θ, while in the second version **M2**, there is a single target surface Y_s, and the dependence on θ is only in the lamination.

Let $\mathbf{TD}_{X,\Phi}$ be the Teichmüller disk determined by X and Φ; i.e. the complex line in $\mathcal{T}(S)$ comprising surfaces whose Teichmüller differentials from a given base point X are all complex multiples of Φ on X. Let

$$h_a:\mathcal{QT}(S)\longrightarrow\mathcal{QT}(S)$$

be the horocyclic flow on the quadratic differential bundle $\mathcal{QT}(S)\to\mathcal{T}(S)$ corresponding to the lower triangular matrix

$$\left\{\begin{pmatrix}1 & 0\\ a & 1\end{pmatrix}: a\in\mathbb{R}\right\}.$$

The horocycle flow acts in the standard way on $\mathbb{R}^2$ once we define charts from the Riemann surface to $\mathbb{R}^2$ by using natural coordinates. Given a Riemann surface X and a holomorphic quadratic differential Φ on X, we choose local coordinates

$$z=x+iy$$

so that $\Phi=dz^2$. With this coordinate, the Teichmüller ray $\mathbf{TR}_{X,\Phi}$ may be interpreted as a family of Riemann surfaces underlying the quadratic differentials

$$\Phi_s:=(s^{1/2}\,dx+s^{-1/2}\,dy)^2.$$

For $a\in\mathbb{R}$, we denote by $h_a(\mathbf{TR}_{X,\Phi})$ the family of Riemann surfaces underlying $h_a(\Phi_s)$ for $s\geqslant 1$. Let

$$\mathbf{TD}^h_{X,\Phi}:=\bigcup_{a\in\mathbb{R}} h_a(\mathbf{TR}_{X,\Phi})$$

be the Teichmüller horodisk, which is the union of the horocyclic translates of $\mathbf{TR}_{X,\Phi}$.

THEOREM 10.1. *$\mathbf{HD}_{X,\Phi,s}$ locally uniformly converges to the Teichmüller disk $\mathbf{TD}_{X,\Phi}$. $\widehat{\mathbf{HD}}_{X,\Phi,s}$ locally uniformly converges to the Teichmüller horodisk $\mathbf{TD}^h_{X,\Phi}$.*

Before proving Theorem 10.1, we state a lemma we will need, deferring its proof to the end of this section.

LEMMA 10.2. *Let $Y_s\in\mathbf{HR}_{X,\Phi}$ and let λ_θ be the horizontal foliation of $e^{2i\theta}\Phi$. Let $\delta>0$ be a constant which is smaller than half of the shortest distance between zeros of Φ. Then, there exists $s_0>0$, which depends only on δ, such that, for $s>s_0$, we have*

$$\ell_{Y_s}(\lambda_\theta)\leqslant 2\sqrt{s}\,i(\lambda_{\pi/2},\lambda_\theta)+96(g-1)\delta^2\sqrt{s}+\|\Phi\|.$$

Proof of Theorem 10.1. The proof is almost the same as the proof of Theorem 9.1.

(1) Convergence of $\mathbf{HD}_{X,\Phi,s}$. Let $X_{s,t,\theta}$ be the Riemann surface such that the maximal stretch lamination of the harmonic map $X_{s,t,\theta}\to Y_{s,\theta}$ is $s^{1/2}t\lambda_\theta$. Let $\Phi_{s,t,\theta}$ be the Hopf differential of $X_{s,t,\theta}\to Y_{s,\theta}$. Then, the Hopf differential of $\widetilde{X}_{s,t,\theta}\to T_{\sqrt{s}t,\theta}$ is

$-\tilde{\Phi}_{s,t,\theta}$, the lift of $-\Phi_{s,t,\theta}$. Therefore,

$$\begin{aligned}
&E(X_{s,t,\theta}, T_{\lambda_{\theta+\pi/2}} \times T_{t\lambda_\theta}) \\
&\quad = E(X_{s,t,\theta}, T_{\lambda_{\theta+\pi/2}}) + E(X_{s,t,\theta}, T_{t\lambda_\theta}) \\
&\quad \leqslant E(X_{s,t,\theta}, s^{-1}Y_{s,\theta}) + E(X_{s,t,\theta}, T_{t\lambda_\theta}) && \text{(by Lemma 9.3)} \\
&\quad = s^{-1}E(X_{s,t,\theta}, Y_{s,\theta}) + s^{-1}E(X_{s,t,\theta}, T_{\sqrt{s}t\lambda_\theta}) \\
&\quad = s^{-1}E(X_{s,t,\theta}, Y_{s,\theta}) + 2s^{-1}\|\Phi_{s,t,\theta}\| \\
&\quad \leqslant s^{-1}(\ell_{Y_{s,\theta}}(\sqrt{s}t\lambda_\theta)+C) + s^{-1}(\ell_{Y_{s,\theta}}(\sqrt{s}t\lambda_\theta)+C) && \text{(by Lemma 3.12)} \\
&\quad = 2s^{-1}t\ell_{Y_{s,\theta}}(\sqrt{s}\lambda_\theta) + 2s^{-1}C \\
&\quad \leqslant 2s^{-1}t(2\|\Phi_{1,s,\theta}\|+C) + 2s^{-1}C && \text{(by Lemma 3.12)} \\
&\quad = 4t\|\Phi_{1,1,\theta}\| + 2s^{-1}C(t+1) \\
&\quad = 4t\|\Phi\| + 2s^{-1}C(t+1).
\end{aligned}$$

Therefore,

$$\lim_{s\to\infty} E(X_{s,t,\theta}, T_{\lambda_{\theta+\pi/2}} \times T_{t\lambda_\theta}) \leqslant 4t\|\Phi\| = 4i(\lambda_{\theta+\pi/2}, t\lambda_\theta).$$

It follows from Lemma 9.2 that $X_{s,t,\theta}$ converges to the Riemann surface $X_{\infty,t,\theta} \in \mathbf{TD}_{X,\Phi}$ underlying the quadratic differential $\Psi_{t,\theta}$ whose horizontal and vertical measured foliations are $t\lambda_\theta$ and $\lambda_{\theta+\pi/2}$.

By considering a function similar to the one defined in (9.2), we see that, for any fixed $t>0$, the convergence is uniform in $(t,\theta) \in [1,t_0] \times [0,2\pi]$.

(2) Convergence of $\widehat{\mathbf{HD}}_{X,\Phi,s}$. Let $\widehat{X}_{s,t,\theta}$ be the Riemann surface such that the maximal stretch lamination of the harmonic map $\widehat{X}_{s,t,\theta} \to Y_s$ is $s^{1/2}t\lambda_\theta$. Let $\hat{\Phi}_{s,t,\theta}$ be the Hopf differential of $X_{s,t,\theta} \to Y_s$. Then,

$$\begin{aligned}
&E(\widehat{X}_{s,t,\theta}, T_{\lambda_{\pi/2}} \times T_{t\lambda_\theta}) \\
&\quad = E(\widehat{X}_{s,t,\theta}, T_{\lambda_{\pi/2}}) + E(\widehat{X}_{s,t,\theta}, T_{t\lambda_\theta}) \\
&\quad \leqslant E(\widehat{X}_{s,t,\theta}, s^{-1}Y_s) + E(X_{s,t,\theta}, T_{t\lambda_\theta}) && \text{(by Lemma 9.3)} \\
&\quad = s^{-1}E(X_{s,t,\theta}, Y_s) + s^{-1}E(X_{s,t,\theta}, T_{\sqrt{s}t\lambda_\theta}) \\
&\quad = s^{-1}E(X_{s,t,\theta}, Y_s) + 2s^{-1}\|\hat{\Phi}_{s,t,\theta}\| \\
&\quad \leqslant s^{-1}(\ell_{Y_s}(\sqrt{s}t\lambda_\theta)+C) + s^{-1}(\ell_{Y_s}(\sqrt{s}t\lambda_\theta)+C) && \text{(by Lemma 3.12)} \\
&\quad = 2s^{-1/2}t\ell_{Y_s}(\lambda_\theta) + 2s^{-1}C(t+1).
\end{aligned}$$

Combined with Lemma 10.2, this implies that for any δ which is smaller than half of the shortest distance between zeros of Φ, there exists $s_0(\delta)>0$, which depends only on δ, such that for $s>s_0$, we have

$$E(\widehat{X}_{s,t,\theta}, T_{\lambda_{\pi/2}}\times T_{t\lambda_\theta}) \leqslant 4i(\lambda_{\pi/2}, t\lambda_\theta)+192(g-1)\delta^2 t+2ts^{-1/2}\|\Phi\|+2s^{-1}C(t+1).$$

Combining Lemma 9.2 and the arbitrariness of δ, we see that, as $s\to\infty$, we have that $E(\widehat{X}_{s,t,\theta}, T_{\lambda_{\pi/2}}\times T_{t\lambda_\theta})$ converges to $4i(\lambda_{\pi/2}, t\lambda_\theta)$ uniformly in $(t,\theta)\in[1,t_0]\times\left(-\frac{1}{2}\pi, \frac{1}{2}\pi\right)$. Let $\widehat{X}_{\infty,t,\theta}\in\mathbf{TD}^h_{X,\Phi}$ be the Riemann surface underlying the quadratic differential $\hat{\Phi}_{t,\theta}$ whose horizontal and vertical measured foliations are $t\lambda_\theta$ and $\lambda_{\pi/2}$. It then follows from Lemma 9.2 that $\widehat{X}_{s,t,\theta}$ converge to $\widehat{X}_{\infty,t,\theta}\in\mathbf{TD}^h_{X,\Phi}$. By considering a function similar to the one defined in (9.2), we see that, for any fixed $t_0>0$, the convergence is uniform in $(t,\theta)\in[1,t_0]\times\left(-\frac{1}{2}\pi, \frac{1}{2}\pi\right)$. □

Proof of Lemma 10.2. Let $\delta>0$ be a fixed constant which is smaller than half of the shortest distance between zeros of Φ. For a zero z_j of $e^{2i\theta}\Phi$ which is of order n_j, let $\mathscr{P}_j$ be a horizontal/vertical $(2n_j)$-polygon of $e^{2i\theta}\Phi$ around z_j such that every horizontal segment and every vertical segment of $\partial\mathscr{P}$ have the same $|\Phi|$-length 2δ. In particular, the $|\Phi|$-distance from z_j to $\partial\mathscr{P}_j$ is δ. Next, we decompose the horizontal foliation F_θ of $e^{2i\theta}\Phi$ outside the union $\bigcup_j \mathscr{P}_j$ into rectangles $\mathscr{R}_i$. Therefore,

$$F_\theta = \left(\bigcup_i (F_\theta\cap\mathscr{R}_i)\right)\cup\left(\bigcup_j (F_\theta\cap\mathscr{P}_j)\right).$$

The hyperbolic length of leaves of $(F_\theta\cap\mathscr{P}_j)$ on Y_s is not convenient to estimate. To overcome this, we need a modification. Notice that the critical leaves of $F_\theta\cap\mathscr{P}_j$ decompose $\mathscr{P}_j$ into several rectangles $\{R_{jk}\}_{1\leqslant k\leqslant n_j}$, where n_j is the order of the zero z_j of Φ. We homotope (relative to its endpoints) each non-critical leaf L of $F_\theta\cap\mathscr{P}_j$ to a curve L' which is contained in the boundary $\partial R_{jk}\cap\partial\mathscr{P}_j$. Equivalently, $L\cup L'$ is the boundary of the component of $R_{jk}\setminus L$ whose closure does not contain the zero z_j of Φ. Then, the lengths of L' and $\partial R_{ji}\cap\partial\mathscr{P}_j$ satisfy

$$\mathrm{Length}_{Y_s}(L')\leqslant \mathrm{Length}_{Y_s}(\partial R_{jk}\cap\partial\mathscr{P}_j),$$

where we use the fact that $L'\subset\partial R_{ji}\cap\partial\mathscr{P}_i$. Let $\mathrm{Length}_{Y_s}(F_\theta\cap R_{jk})$ be the hyperbolic length of the leaves of the foliation F_θ restricted to R_{jk}, integrated over the induced transverse measure of the family of leaves. Then, we have

$$\mathrm{Length}_{Y_s}(F_\theta\cap R_{jk})\leqslant \delta\,\mathrm{Length}_{Y_s}(\partial R_{jk}\cap\partial\mathscr{P}_j). \tag{10.1}$$

Let $\mathrm{Length}_{Y_s}(F_\theta\cap\mathscr{R}_i)$ be the hyperbolic length of the leaves of the foliation F_θ restricted to $\mathscr{R}_i$, integrated over the induced transverse measure of the family of leaves. By definition, $\ell_{Y_s}(\lambda_\theta)$ is the Y_s-length of geodesic representatives of leaves of F_θ, integrated over the induced transverse measure of family of leaves. Hence,

$$\begin{aligned}\ell_{Y_s}(\lambda_\theta) &\leqslant \sum_i \mathrm{Length}_{Y_s}(F_\theta\cap\mathscr{R}_i)+\sum_j \mathrm{Length}_{Y_s}(F_\theta\cap\mathscr{P}_j)\\ &= \sum_i \mathrm{Length}_{Y_s}(F_\theta\cap\mathscr{R}_i)+\sum_j \sum_{1\leqslant k\leqslant n_j} \mathrm{Length}_{Y_s}(F_\theta\cap R_{jk})\\ &\leqslant \sum_i \mathrm{Length}_{Y_s}(F_\theta\cap\mathscr{R}_i)+\sum_j \delta \sum_{1\leqslant k\leqslant n_j} \mathrm{Length}_{Y_s}(\partial R_{jk}\cap\partial\mathscr{P}_j) \qquad \text{(by (10.1))}\\ &= \sum_i \mathrm{Length}_{Y_s}(F_\theta\cap\mathscr{R}_i)+\delta\sum_j \mathrm{Length}_{Y_s}(\partial\mathscr{P}_j). \end{aligned} \tag{10.2}$$

Recall that with respect to the canonical coordinate of Φ, the hyperbolic metric on Y_s can be expressed as

$$\begin{aligned} ds_Y^2 &= 2s(\cosh\mathscr{G}(z,s)+1)\,dx^2+2s(\cosh\mathscr{G}(z,s)-1)\,dy^2\\ &= 4s\,dx^2+2s(\cosh\mathscr{G}(z,s)-1)(dx^2+dy^2), \end{aligned} \tag{10.3}$$

where $\mathscr{G}(z,s)=\log(1/|\nu(z,s)|)$ and $\nu(z,s)$ is the Beltrami differential of the harmonic map $X\to Y_s$. By Lemma 3.5, we see that there exists $\mathbf{s_0}>1$, such that for every $s>\mathbf{s_0}$ and every $z\in\overline{X\setminus(\bigcup_j \mathscr{P}_j)}$, we have

$$2s(\cosh\mathscr{G}(z,s)-1) < 1. \tag{10.4}$$

Let w_i and h_i be respectively the horizontal width and vertical height of $\mathscr{R}_i$ with respect to $e^{2i\theta}\Phi$. Combining (10.3) and (10.4), we see that

$$\begin{aligned} \mathrm{Length}_{Y_s}(F_\theta\cap\mathscr{R}_i) &\leqslant (2\sqrt{s}w_i\cos\theta+w_i)h_i\\ &= 2\sqrt{s}i(F_{\pi/2}\cap\mathscr{R}_i, F_\theta\cap\mathscr{R}_i)+w_ih_i\\ &= 2\sqrt{s}i(F_{\pi/2}\cap\mathscr{R}_i, F_\theta\cap\mathscr{R}_i)+\|\Phi\|_{\mathscr{R}_i}. \end{aligned} \tag{10.5}$$

Recall that $\partial\mathscr{P}_j$ consists of n_j horizontal arcs and n_j vertical arcs with respect to $e^{2i\theta}\Phi$. Moreover, every such arc has the same $|\Phi|$-length δ. Hence, by (10.3) and (10.4),

$$\mathrm{Length}_{Y_s}(\partial\mathscr{P}_j) \leqslant n_j(2\sqrt{s}\delta\cos\theta+2\delta)+n_j(2\sqrt{s}\delta\sin\theta+2\delta) \leqslant 8n_j\sqrt{s}\delta. \tag{10.6}$$

Combining (10.2), (10.5) and (10.6), we have

$$\begin{aligned} \ell_{Y_s}(\lambda_\theta) &\leqslant 2\sqrt{s}i(F_{\pi/2}, F_\theta)+\|\Phi\|+\sum_j 8n_j\delta^2\sqrt{s}\\ &\leqslant 2\sqrt{s}i(F_{\pi/2}, F_\theta)+\|\Phi\|+96(g-1)\delta^2\sqrt{s}, \end{aligned}$$

where the last inequality follows from the fact $\sum_j n_j\leqslant 12(g-1)$. $\square$

11. Convergence to stretch-earthquake disks

We recall some basic facts about earthquake deformations. For more details, we refer to [48, §II] (see also [49] and [62]). Let $\mu\alpha$ be a weighted simple closed geodesic on X. The *time-s earthquake* $\mathcal{E}^s_{\mu\alpha}(X)$ of X is defined to be the hyperbolic surface obtained from X by twisting left by a distance $s\mu\ell_X(\alpha)$ along α. The weight μ determines the speed of twisting. This construction extends to general measured laminations as follows. For a measured geodesic lamination λ, let $\mu_i\alpha_i$ be a sequence of weighted simple closed geodesics that converges to λ in $\mathcal{ML}(S)$. The *time-s earthquake* $\mathcal{E}^s_\lambda(X)$ of X is defined as the limit surface of $\mathcal{E}^s_{\mu_i\alpha_i}(X)$. It is a non-trivial fact that the limit exists and is independent of the choice of the approximating sequence $\mu_i\alpha_i$ ([48, Corollary 2.5]).

Definition 11.1. (Stretch-earthquake disk) Let $Y\in\mathcal{T}(S)$ be a hyperbolic surface and $\lambda\in\mathcal{ML}(S)$ a measured foliation/lamination. Let $\mathrm{PSL}_{Y,\lambda}$ be the piecewise harmonic stretch line obtained from Theorem 1.1 as the limit of harmonic map rays $\mathbf{HR}_{X_t,Y}$ where X_t diverges along the harmonic map dual ray $\mathbf{hr}_{Y,\lambda}$. Let $\mathcal{E}^s_\lambda(Y)$ be the surface obtained from Y by acting by a time s (left) earthquake along λ. Define the stretch-earthquake disk $\mathbf{SED}(Y,\lambda)$ of (Y,λ) to be the set

$$\bigcup_{-\infty<s<+\infty} \mathrm{PSL}_{\mathcal{E}^s_\lambda(Y),\lambda}(0,+\infty).$$

Definition 11.2. (Hopf differential disk) Define the Hopf differential disk $\mathbf{HDD}(X,\Phi)$ of (X,Φ) to be the set

$$\bigcup_{-\pi\leqslant\theta\leqslant\pi} \mathbf{HR}_{X,e^{i\theta}\Phi}(0,\infty).$$

Let $X_t\in\mathbf{hr}_{Y,\lambda}$ be the Riemann surface such that the horizontal foliation of

$$\mathrm{Hopf}(X_t,Y)=\Phi_t$$

is $t\lambda$. Let $Y(t,r,s)$ be the hyperbolic surface such that

$$\mathrm{Hopf}(X_t,Y(t,r,s))=re^{is/2t}\Phi_t \quad\text{and}\quad Y_r=\mathrm{PSL}_{Y,\lambda}(r)\in\mathrm{PSL}_{Y,\lambda}.$$

The goal of this section is to prove Theorem 1.14, which we restate here.

THEOREM 11.3. *Let $Y\in\mathcal{T}(S)$ be a hyperbolic surface and $\lambda\in\mathcal{ML}(S)$ a measured foliation/lamination. Let $X_t\in\mathbf{hr}_{Y,\lambda}$ be as described above.*

Then, the family of Hopf differential disks $(\mathbf{HDD}(X_t,\Phi_t),Y)$ with base point Y locally uniformly converges to the stretch-earthquake disk $(\mathbf{SED}(Y,\lambda),Y)$ with base point Y. Namely, for any prescribed $\mathbf{s}>0$ and $0<\mathbf{r}<\mathbf{r}'$, the point $Y(t,r,s)$ converges to $\mathcal{E}^s_\lambda(Y_r)$, uniformly in $(r,s)\in[\mathbf{r},\mathbf{r}']\times[-\mathbf{s},\mathbf{s}]$, as $t\to\infty$.

Recall that the family of harmonic maps $f_t\colon X_t \to Y$ converges to the harmonic map $f_\infty\colon X_\infty \to Y$ with $\mathrm{Hopf}(f_\infty) = \Phi_\infty$, where Φ_∞ is the union of half-infinite cylinders and half-planes (Lemma 8.3).

The proof of Theorem 11.3 relies on a generalization of shearing coordinates of $\mathcal{T}(S)$, namely the shear-shape coordinates of $\mathcal{T}(S)$ developed by Calderon–Farre ([11]).

11.1. Shearing

In order to glue crowned hyperbolic surfaces, one needs a foliation transverse to the boundary of crowned surfaces. Thurston [90] uses the horocycle foliation (where the crowned hyperbolic surfaces are ideal triangles), Calderon and Farre [11] use the orthogeodesic foliation. Here, we use the extended foliation of the vertical foliation of Hopf differentials as in the construction of piecewise harmonic stretch lines.

In this subsection, we will construct *shear-shape cocycles* of λ using the vertical foliation of Hopf differentials, following the ideas and constructions of Calderon–Farre.

11.1.1. Extended vertical foliations

Let $Y \in \mathcal{T}(S)$ be an arbitrary hyperbolic surface and let λ be a measured geodesic lamination on Y. Let η be the admissible measured foliation (Definition 5.2) on $Y \setminus \lambda$ that comprises half-infinite cylinders foliated by closed leaves parallel to closed geodesic boundary components of $Y \setminus \lambda$ and half-planes foliated by bi-infinite leaves parallel to ideal boundary geodesics of $Y \setminus \lambda$. Note that η is determined by λ. By Theorem 8.7, there exists a unique (possibly disconnected) punctured Riemann surface X_∞, homeomorphic to $Y \setminus \lambda$, and a unique harmonic diffeomorphism $f_\infty\colon X_\infty \to Y \setminus \lambda$, homotopic to the identity map such that the pushforward to $Y \setminus \lambda$ of the horizontal measured foliation of the Hopf differential of f_∞ is measure equivalent to λ. By Lemma 7.7 and Lemma 7.8, the pushforward to $Y \setminus \lambda$ of the vertical foliation of the Hopf differential of f_∞ extends to a unique measured foliation β_Y on Y which is transverse to η and λ and intersects λ orthogonally at every intersection point.

In the remainder of this section, for any $Z \in \mathcal{T}(S)$, whenever we mention η and β on $Z \setminus \lambda$, we mean (unless otherwise stated) their realization on $Z \setminus \lambda$ as the pushforward by f_∞ of the horizontal foliation and vertical foliation of the Hopf differential of f_∞ (where X_∞, f_∞ and Hopf f_∞ refer to Z and λ).

Let $\mathcal{T}(S \setminus \lambda)$ be the Teichmüller space of crowned hyperbolic surfaces homeomorphic to $S \setminus \lambda$. Let $\mathcal{MF}(\lambda) \subset \mathcal{MF}(S)$ be the subset of measured foliations transverse to λ. The construction above defines a map

$$\pi_\lambda: \mathcal{T}(S) \longrightarrow \mathcal{T}(S \setminus \lambda) \times \mathcal{MF}(\lambda) \tag{11.1}$$

which sends $Z \in \mathcal{T}(S)$ to the pair $(Z \setminus \lambda, \beta_Z)$. Notice that β_Z prescribes identifications for gluing both the leaves of $\beta_Z|_{Z\setminus\lambda}$ as well as the components of $Z\setminus\lambda$ across λ. To describe the gluing process, we introduce an analogue of the *shear-shape cocycle* developed by [11] using pointed geodesics.

11.1.2. Dual arc systems

We start with the definition of dual arc systems. Let Y, λ and η be as above. Let $\pi_\lambda(Y) = (Y\setminus\lambda, \beta_Y)$. For simplicity, we denote β_Y by β. Recall that every leaf of $\beta|_{Y\setminus\lambda}$ approaching $\partial Y \subset \lambda$ hits $\partial Y \setminus \lambda$ orthogonally (Lemma 7.8). For every leaf segment e of η that connects a pair of singular points of η (a "*saddle connection of* η"), there exists a strip Strip_e of $\beta|_{Y\setminus\lambda}$ foliated by regular leaves of $\beta|_{Y\setminus\lambda}$ that intersect e orthogonally and approaches a pair of boundary geodesics of $\partial(Y\setminus\lambda)$ orthogonally. Let c_e be the height of Strip_e. Then, by Lemma 7.8, we see that c_e is half the hyperbolic length of the projection of Strip_e to a boundary geodesic of $Y\setminus\lambda$ along leaves of $\beta|_{Y\setminus\lambda}$. Note that any two leaves of $\beta|_{Y\setminus\lambda}$ in Strip_e are homotopic rel $\partial(Y\setminus\lambda)$. Let α_e be the regular leaf of $\beta|_{Y\setminus\lambda}$ that cuts Strip_e into two substrips of equal height. The *dual arc system* $\underline{\alpha}(Y\setminus\lambda)$ of $Y\setminus\lambda$ is defined to be the union $\bigcup_e \alpha_e$, where e ranges over all saddle connections of η. Define the *weighted dual arc system* as the formal sum

$$\underline{A}(Y\setminus\lambda) := \sum_e c_e \alpha_e,$$

where e ranges over all saddle connections of η. Notice that η has finitely many saddle connections. The dual arc system cuts $Y\setminus\lambda$ into pieces, each of which contains exactly one singularity of η (here η and $\beta|_{Y\setminus\lambda}$ have the same set of singularities). Following Caldron–Farre, we call each such piece a *hexagon*, no matter its shape. Let $\mathbf{H}$ be the set of hexagons of $Y\setminus\lambda$.

11.1.3. Pointed geodesics and shear

Let $\widetilde{Z}$, $\widetilde{Y\setminus\lambda}$, $\tilde{\beta}$, $\tilde{\eta}$ and $\widetilde{\mathbf{H}}$ be the lifts of Z, $Y\setminus\lambda$, β, η and $\mathbf{H}$, respectively. We now define a family of base points associated to the boundary leaves of $\tilde{\lambda}$. Let $H_v \in \widetilde{\mathbf{H}}$ be a hexagon with a singular point v of $\widetilde{\beta|_{Y\setminus\lambda}}$. For a boundary leaf g_v of $\tilde{\lambda}$ intersecting ∂H_v, define p_v to be the projection of v to g along the half-infinite critical leaf of $\widetilde{\beta|_{Y\setminus\lambda}}$ that starts at v and approaches g. The pair (g_v, p_v) is called a *pointed geodesic*. For any pair of hexagons $H_v, H_w \in \widetilde{\mathbf{H}}$ belonging to distinct components of $\widetilde{Y \setminus \lambda}$, there is a unique geodesic g_v intersecting ∂H_v that separates v from w. Symmetrically, there is a unique geodesic g_w intersecting ∂H_w that separates w from v. The pair (g_v, p_v) and (g_w, p_w) is said to be *simple* if g_v and g_w are connected by a regular leaf segment ξ_v^w (not unique) of $\tilde{\beta}$ such that for each component Ω (intersecting ξ_v^w) of $\widetilde{Y \setminus \lambda}$, the segment ξ_v^w cuts out a cusp region from Ω. Let q_v (resp. q_w) be the intersection point between g_v (resp. g_w) and ξ_v^w. The *shear* $\sigma_\lambda(Y)(v, w)$ among H_v and H_w is defined to be the sum of the signed distance from $p_v \in g_v$ to $q_v \in g_v$ along g_v and the signed distance from $p_w \in g_w$ to $q_w \in g_w$ along g_w, with respect to the orientations of g_v and g_w induced from the orientation of $\widetilde{Y \setminus \lambda}$.

11.1.4. Weighted system

Let τ be an ε *train track* of λ, that is, a train track which carries λ and which is contained in the ε-neighbourhood of λ such that $Y \setminus \lambda$ and $Y \setminus \tau$ have the same topological type. The orientation of Y induces an orientation for each component of $Y \setminus \lambda$. A *standard smoothing* train track $\tau_{\underline{\alpha}}$ of $\tau \cup \underline{\alpha}(Y \setminus \lambda)$ is a smoothing at each intersection point $\tau \cap \underline{\alpha}(Y \setminus \lambda)$ so that the incoming tangent vector corresponding to $\underline{\alpha}(Y \setminus \lambda)$ points in the positive direction with respect to the boundary orientation of $Y \setminus \lambda$ (see [11, p. 2049, Figure 12]). The shears among simple pairs of pointed geodesics defined as above define a *weighted system* $\mathbf{w}(Y)$ of $\tau_{\underline{\alpha}}$ as follows.

- To each branch corresponding to a component α of $\underline{\alpha}$, assign the weight to be the height of the strip of $\beta|_{Y \setminus \lambda}$ foliated by leaves parallel to α.
- To each branch b which is not a component of $\underline{\alpha}$, choose a lift $\tilde{b}$ of b. Let $H_v, H_w \in \widetilde{\mathbf{H}}$ be the pair of hexagons adjacent to $\tilde{b}$ and set the weight to be $\sigma_\lambda(Y)(v, w)$.

The proof that $\mathbf{w}(Y)$ satisfies the switch condition of $\tau_{\underline{\alpha}}$ is identical to the proof of [11, Lemma 13.6]. The advantage of working with weighted systems is that each weighted system is determined by the weight of branches, whose number is uniformly bounded in terms of the topology of Y. This finiteness will be used later in the proof of locally uniform convergence.

11.1.5. Shear-shape cocycle

By [11, Propositions 7.13 and 9.5], each weighted system of $\tau_{\underline{\alpha}}$ defines a shear-shape cocycle $(\underline{A}(Y \setminus \lambda), \sigma)$ in the sense of [11, Definition 7.11], where σ is a function which assigns to every arc k transverse to λ and disjoint from $\underline{\alpha} := \bigcup \alpha_i$ a real number $\sigma(k)$, satisfying the following axioms.

(1) (Support) If k does not intersect λ, then $\sigma(k)=0$.

(2) (Transverse invariance) If k and k' are isotopic transverse to λ and disjoint from $\underline{\alpha}$ then $\sigma(k)=\sigma(k')$.

(3) (Finite additivity) If $k=k_1\cup k_2$ where k_i have disjoint interiors then

$$\sigma(k)=\sigma(k_1)+\sigma(k_2).$$

(4) ($\underline{A}$-compatibility) Suppose that k is transverse to λ and isotopic rel endpoints to some arc which may be written as $t_i\cup l$, where t_i is an arc which meets α_i exactly once and is disjoint from $\lambda\cup\underline{\alpha}\setminus\{\alpha_i\}$, and l is an arc transverse to λ and disjoint from $\underline{\alpha}$. Then, the loop $k\cup t_i\cup l$ encircles a unique point p of $\lambda\cap\underline{\alpha}$, and

$$\sigma(k)=\sigma(l)+\varepsilon c_i,$$

where ε denotes the winding number of $k\cup t_i\cup l$ about p, where the loop is oriented so that the edges are traversed k then t_i then l (see [11, p. 2038, Figure 9]).

Applying verbatim the argument in the proof of [11, Lemma 13.9], we see that the resulting shear-shape cocycle is independent of the choice of $\tau_{\underline{\alpha}}$. This defines a map

$$\begin{aligned}\Pi_\lambda\colon \mathcal{T}(S) &\longrightarrow \mathcal{T}(S\setminus\lambda)\times\mathcal{SH}(\lambda),\\ Y &\longmapsto (Y\setminus\lambda, (\underline{A}(Y\setminus\lambda), \sigma_\lambda(Y))),\end{aligned}$$

where $\mathcal{SH}(\lambda)$ is the space of shear-shape cocycles of λ.

Proposition 11.4. *The map*

$$\Pi_\lambda\colon \mathcal{T}(S) \longrightarrow \mathcal{T}(S\setminus\lambda)\times\mathcal{SH}(\lambda)$$

is homeomorphic onto its image.

Proof. We first show that Π_λ is injective. Suppose that $Z, Z'\in\mathcal{T}(S)$ are such that $\Pi_\lambda(Z')=\Pi_\lambda(Z)$. Then, by the computation of monodromy in the proof of Theorem 1.5, we see that Z and Z' have the same monodromy group, so they are equal. Another proof is to construct an equivariant isometry from the universal cover $\widetilde{Z}$ to $\widetilde{Z}'$ by following the argument as in the proof of [7, Lemma 11] or [11, Proposition 13.12] (except for the second paragraph there because we include $\mathcal{T}(S\setminus\lambda)$ in the codomain of the map σ_λ).

Next, note that Π_λ is clearly continuous. For the inverse map $\Pi_\lambda^{-1}\colon \Pi_\lambda(\mathcal{T}(S))\to\mathcal{T}(S)$ from the image $\Pi_\lambda(\mathcal{T}(S))$, we reconstruct the hyperbolic surface Z from the components $Z\setminus\lambda\in\mathcal{T}(S\setminus\lambda)$ of the complement of λ and the shear-shape cocycle

$$(\underline{A}(Z\setminus\lambda), \sigma_\lambda(Z))\in\mathcal{SH}(\lambda)$$

via a canonical construction: as mentioned in §11.1.1, the crowned surface $Z\backslash\lambda$ and the admissible foliation η (determined by λ) on $Z\backslash\lambda$ gives a unique vertical foliation $\beta|_{Z\backslash\lambda}$ on $Z\backslash\lambda$. Then, the function $\sigma_\lambda(Z)$ provides the data for gluing the components of $\beta|_{Z\backslash\lambda}$ across λ to get a unique measured foliation β_Z on the whole surface; using that extension of the foliation β_Z, we then glue together components of $Z\backslash\lambda$ in a way similar to the method we used in the construction of a piecewise harmonic stretch line (cf. §7.5). The computation of monodromy in the proof of Theorem 1.5 then implies that the inverse map $\Pi_\lambda^{-1}: \Pi_\lambda(\mathcal{T}(S)) \to \mathcal{T}(S)$ is also continuous. Therefore, the map Π_λ is homeomorphic onto its image $\Pi_\lambda(\mathcal{T}(S))$. □

Let

$$\begin{aligned}\hat{\sigma}_\lambda: \mathcal{T}(S) &\longrightarrow \mathcal{SH}(\lambda),\\ Y &\longmapsto (\underline{A}(Y\backslash\lambda), \sigma_\lambda(Y)),\end{aligned}$$

be the composed map

$$\mathcal{T}(S) \longrightarrow \mathcal{T}(S\backslash\lambda) \times \mathcal{SH}(\lambda) \longrightarrow \mathcal{SH}(\lambda),$$

where the second map is the projection map.

11.2. Proof of Theorem 11.3

In view of the map $\Pi_\lambda: \mathcal{T}(S) \to \mathcal{T}(S\backslash\lambda) \times \mathcal{SH}(\lambda)$, any limit of $Y(t,r,s)$, as $t\to\infty$, is described by a crowned surface and a shear-shape cocycle of λ which encodes the shearing between the components of the limit crowned surface.

Lemma 11.5. (Shape convergence) *$Y(t,r,s)\backslash\lambda$ converges to $Y_r\backslash\lambda$, locally uniformly in $(r,s)\in[\mathbf{r},\mathbf{r}']\times[-\mathbf{s},\mathbf{s}]$, as $t\to\infty$.*

Proof. Recall that

$$\operatorname{Hopf}(X_t, Y(t,r,s)) = e^{is/2t} \operatorname{Hopf}(X_t, Y(t,r,0)) = re^{is/2t}\Phi_t,$$

where $\Phi_t = \operatorname{Hopf}(X_t, Y)$. By (3.6) and (3.9), we see that the energy density $\mathbf{e}$, which is defined to be $\mathcal{H}+\mathcal{L}$ in the notation introduced in (3.6) and (3.9), satisfies

$$\mathbf{e}(X_t, Y(t,r,s)) = \mathbf{e}(X_t, Y(t,r,0)).$$

The pullback of the hyperbolic metric $\rho(t,r,s)$ of $Y(t,r,s)$ to X_t via the harmonic map $f_{t,r,s}: X_t \to Y(t,r,s)$ is

$$(f_{t,r,s})^*\rho(t,r,s) = re^{is/2t}\Phi_t + \overline{re^{is/2t}\Phi_t} + \sigma_t \mathbf{e}(X_t, Y(t,r,s)),$$

where σ_t is the hyperbolic metric on X_t. This gives

$$(f_{t,r,s})^*\rho(t,r,s)-(f_{t,r,0})^*\rho(t,r,0)=r(e^{is/2t}-1)\Phi_t+\overline{r(e^{is/2t}-1)\Phi_t}.$$

Combining this with the fact that Φ_t converges to Φ_∞, uniformly on compact subsets of $X_\infty \overset{\text{homeo}}{\sim} Y\setminus\lambda$, as $t\to\infty$, we see that on any compact subset of $Y\setminus\lambda$, we have that $Y(t,r,s)$ converges to Y_r uniformly in $(r,s)\in[\mathbf{r},\mathbf{r}']\times[-\mathbf{s},\mathbf{s}]$ as $t\to\infty$. Since any crowned hyperbolic surface is determined by its truncation (i.e. removing a small neighbourhood for each cusp) and each truncation is of arbitrary small distance to compact subsets of the crowned surface, so $Y(t,r,s)\setminus\lambda$ itself converges to $Y_r\setminus\lambda$ uniformly in $(r,s)\in[\mathbf{r},\mathbf{r}']\times[-\mathbf{s},\mathbf{s}]$ as $t\to\infty$. □

LEMMA 11.6. (Shear-shape convergence) *The shear-shape cocycle $\hat\sigma_\lambda(Y(t,r,s))$ converges to $\hat\sigma_\lambda(\mathcal{E}^t_\lambda(Y_r))$ uniformly in $(r,s)\in[\mathbf{r},\mathbf{r}']\times[-\mathbf{s},\mathbf{s}]$ as $t\to\infty$.*

Proof. Notice that the set of dual arc systems $\{\underline{\alpha}(Y_r\setminus\lambda): r\in[\mathbf{r},\mathbf{r}']\}$ is a finite set. The locally uniform convergence of $Y(t,r,s)\setminus\lambda$ to $Y_r\setminus\lambda$ in $(r,s)\in[\mathbf{r},\mathbf{r}']\times[-\mathbf{s},\mathbf{s}]$ as $t\to\infty$ then implies that

$$\{\underline{\alpha}(Y(t,r,s)\setminus\lambda): t>T_0, r\in[\mathbf{r},\mathbf{r}'], s\in[-\mathbf{s},\mathbf{s}]\}$$

is also finite, and that the weighted dual arc system $\underline{A}(Y(t,r,s)\setminus\lambda)$ uniformly converges to the weighted dual arc system $\underline{A}(Y_r\setminus\lambda)$ for $r\in[\mathbf{r},\mathbf{r}']$. To prove the lemma, we may assume that the set above contains exactly one element, say the dual arc system $\underline{\alpha}$.

Next, we calculate the transverse cocycle $\sigma_\lambda(Y(t,r,s))$ of $Y(t,r,s)$ with respect to λ, using the train-track coordinates of transverse cocycles described in §11.1.4. We begin with a preliminary discussion in the setting where $s=0$, and then extend the analysis to the case where $s\neq 0$.

For the analysis, we fix a convenient choice of scale R_t for the Minsky region $\mathscr{P}_{R_t}(\Phi_t)$. In particular, let $R_t:=t^{1/4}$. By Lemma 8.5, for any $\varepsilon>0$, there exists $T_0>0$ such that, for any $t>T_0$, the train track τ_{t,R_t} in Y corresponding to $X_t\setminus\mathscr{P}_{R_t}(\Phi_t)$ carries λ and is contained in the ε neighbourhood of λ. Lift everything to the universal cover. Together with the dual arc system $\underline{\alpha}$, this train-track gives a finite coordinatization of $\sigma_\lambda(Y(t,r,s))$, which consists of a finite collection of simple pairs of (sufficiently close) pointed geodesics, denoted by $\mathscr{G}$. Let $((g_v,p_v),(g_w,p_w))\in\mathscr{G}$ be such a simple pair of pointed geodesics with $p_v\in g_v$ and $p_w\in g_w$ being the base points. Let β be the extended vertical foliation on Y of Φ_∞ (which is also the measured foliation in $\mathcal{MF}_\eta(\lambda)$ given by the map $\pi_\lambda(Y)$ in (11.1)). Since (g_v,p_v) and (g_w,p_w) are (sufficiently close) simple pairs, there exists a curve k_{vw} on $\widetilde{Y}$ that is a concatenation of a (horizontal) leaf segment k^1_{vw} of g_v from p_v to some point q_v, a (vertical) leaf segment ξ^w_v of β from q_v to some point

$q_w \in g_w$, and a (horizontal) leaf segment k^2_{vw} of g_w from q_w to p_w. Hence, by definition, we have

$$\sigma_\lambda(Y_r)((g_v,p_v),(g_w,p_w)) = \sigma_\lambda(Y_r)(k_{vw}) = 2\sqrt{r}\sum_{j=1}^{2}\varepsilon_j \cdot i(k^j_{vw},\beta),$$

where $\varepsilon_j \in \{\pm 1\}$ depends on the oriented distance along g_v from p_v to q_v (resp. along g_w from p_w to q_w).

Now, by Theorem 3.9 and Lemma 8.5, we could exponentially closely approximate (g_v,p_v) and (g_w,p_w) by a family of pointed Φ_t-horizontal leaves $(g_v(t),p_v(t))$ and $(g_w(t),p_w(t))$, and exponentially closely approximate k_{vw} by the image of a family of Φ_t-polygonal curves $k_{vw}(t)$ on $\widetilde{Y}$ that is a concatenation of a horizontal leaf segment $k^1_{vw}(t)$ of $g_v(t)$ from $p_v(t)$ to some point $q_v(t)$, a vertical leaf segment $\xi^w_v(t)$ from $q_v(t)$ to some point $g_w(t)$, and a horizontal leaf segment $k^2_{vw}(t)$ of $g_w(t)$ from $q_w(t)$ to $p_w(t)$; here we may choose $k_{vw}(t)$ in the complement of $\mathscr{P}_{R_t}(X_t)$ as λ is in the complement of the image of $\mathscr{P}_{R_t}(X_t)$. Then, from Theorem 3.9, because the leaves of λ are well-approximated by images of Φ_t-horizontal arcs, with distances along λ well-approximated by $4\Phi_t$-horizontal lengths (outside of the polygonal region $\mathscr{P}_{R_t}(X_t)$), we may estimate that

$$\begin{aligned}
&\sigma_\lambda(Y(t,r,0))((g_v,p_v),(g_w,p_w))\\
&\quad = \sigma_\lambda(Y(t,r,0)(k_{vw})\\
&\quad = 2\sqrt{r}\sum_{j=1}^{2}\varepsilon_j \cdot i(k^j_{vw}(t),\operatorname{Vert}(\Phi_t))(1+O(\exp(-brR_t)))\\
&\quad \to 2\sqrt{r}\sum_{j=1}^{2}\varepsilon_j \cdot i(k^j_{vw},\beta) \qquad\qquad (\text{as } t\to\infty)\\
&\quad = \sigma_\lambda(Y_r)((g_v,p_v),(g_w,p_w)),
\end{aligned}$$

where b in the third line above is the constant from Theorem 3.9 and $\varepsilon_j \in \{\pm 1\}$ depends on the oriented distance along g_v from p_v to q_v (resp. along g_w from p_w to q_w). The final equality is by substituting the results of the previous displayed computation.

We now consider the effect of a non-zero "rotation factor" $s \neq 0$. By Lemma 8.5, for t sufficiently large, the image of $X_t \setminus \mathscr{P}_{2R_t}(\Phi_t)$ under the harmonic map $X_t \to Y(t,r,s)$ is a thickened train track that carries λ and is contained in the ε neighbourhood of λ (here we use the fact $\operatorname{Hor}(\Phi_t)=t\lambda$). Notice that for any fixed $\mathbf{s}>0$ (and as usual, for t sufficiently large), the complement $X_t \setminus \mathscr{P}_{R_t}(e^{is/2t}\Phi_t)$ contains $X_t \setminus \mathscr{P}_{2R_t}(\Phi_t)$ for $-\mathbf{s} \leqslant s \leqslant \mathbf{s}$. Hence, the image of $X_t \setminus \mathscr{P}_{R_t}(e^{is/2t}\Phi_t)$ under the harmonic map $X_t \to Y(t,r,s)$, which is contained in the ε neighbourhood of the geodesic lamination $\operatorname{Hor}(e^{is/2t}\Phi_t)$ by

Theorem 3.9, is also a thickened train track carrying λ. Therefore,

$$\begin{aligned}
&\sigma_\lambda(Y(t,r,s))((g_v,p_v),(g_w,p_w))) \\
&\quad = \sigma_\lambda(Y(t,r,s))(k_{vw}) \\
&\quad = 2\sqrt{r}\left(\sum_{j=1}^{2} \varepsilon_j \cdot i(k_{vw}^j(t), \mathrm{Vert}(\Phi_t)) \cos\frac{s}{2t} + i(k_{vw}(t), \mathrm{Hor}(\Phi_t)) \sin\frac{s}{2t}\right) \\
&\qquad\quad \times (1+O(\exp(-brR_t))) \\
&\quad \to 2\sqrt{r}\sum_{j=1}^{2} \varepsilon_j \cdot i(k_{vw}^j, \beta) + s\sqrt{r} i(k_{vw}, \lambda) \qquad\qquad (\text{as } t\to\infty) \\
&\quad = \sigma_\lambda(Y_r)((g_v,p_v),(g_w,p_w)) + s\sqrt{r} i(k_{vw}, \lambda).
\end{aligned}$$

This means that

$$\sigma_\lambda(Y(t,r,s) \to \sigma_\lambda(\mathcal{E}_\lambda^s(Y_r))$$

as $t\to\infty$. The finiteness of the pairs of oriented geodesics then implies that the convergence is locally uniform in $(r,s)\in[\mathbf{r},\mathbf{r}']\times[-\mathbf{s},\mathbf{s}]$. □

Proof of Theorem 11.3. The theorem now follows from Proposition 11.4, and Lemmas 11.5 and 11.6. □

12. Harmonic-stretch lines between hyperbolic surfaces

The goal of this section is, in outline, to find for every ordered pair (Y,Z) of distinct hyperbolic surfaces in $\mathcal{T}(S)$, a unique harmonic stretch line proceeding from Y to Z, i.e. a unique Thurston geodesic proceeding from Y to Z determined only by Y and Z, together with an extra side condition resulting from a requirement of minimizing a variational quantity. The main result is stated in Theorem 1.6 which characterizes those stretch lines in terms of "admissible triples". We give that definition in the first subsection before proceeding to state and prove Theorem 1.6 in the following subsections.

12.1. Harmonic stretch lines and admissible triples

Recall that (Theorem 1.5) a piecewise harmonic stretch line is constructed from a closed hyperbolic surface U, a geodesic lamination λ on U, and a harmonic diffeomorphism $X\to U\setminus\lambda$ from some (possibly disconnected) punctured Riemann surface.

Definition 12.1. We say that a piecewise harmonic stretch line is a *harmonic stretch line*, if it is a limit of a sequence of harmonic map rays. We define a *harmonic stretch ray* from Y to be the restriction of a harmonic stretch line to a half-infinite subsegment that begins at Y. We adopt the notation **HSL** and **HSR**, respectively, often decorated with their defining data.

Let Y and Z be two fixed hyperbolic surfaces in $\mathcal{T}(S)$. Let λ be the maximally stretched chain-recurrent lamination from Y to Z (§2.6.1). Let L be the least Lipschitz constant for Lipschitz maps from Y to Z in the prescribed homotopy class. We will write, for example, $L^{-2}Z$ to indicate the constant curvature metric obtained by scaling the hyperbolic metric Z by the constant L^{-2} (so that the line element is scaled by L^{-1}).

Definition 12.2. A triple (X, f, h) is said to be *admissible* if the following hold:

(i) X is a punctured Riemann surface (possibly disconnected);

(ii) $f\colon X\to Y\setminus\lambda$ and $h\colon X\to L^{-2}Z\setminus\lambda$ are harmonic diffeomorphisms with meromorphic Hopf differentials satisfying $\operatorname{Hopf}(f)=\operatorname{Hopf}(h)$;

(iii) the piecewise harmonic stretch line defined by $f\colon X\to Y\setminus\lambda$ passes through Z;

(iv) the piecewise harmonic stretch line defined by $f\colon X\to Y\setminus\lambda$ is a harmonic stretch line.

Remark 12.3. We emphasize that the final condition in the definition requires (cf. Definition 12.1) that the piecewise harmonic stretch line is a *limit* of harmonic map rays. Note the nuance in the final two conditions: we require the piecewise harmonic stretch line to pass through Y and Z, we require that piecewise harmonic stretch line to be a limit of harmonic map rays, but we do not require the harmonic map ray approximates implied in condition (iv) to pass through both Y and Z.

For the existence of admissible triples, we refer forward to §12.3. By item (b) of Theorem 1.5, we see that $h\circ f^{-1}$ extends to a 1-Lipschitz homeomorphsim from Y to $L^{-2}Z$.

In §13, we will extend this definition to the case where the terminal surface Z is replaced by an $\mathbb{R}$-tree. (Naturally, we will have to lift all of the definitions to spaces on which $\pi_1(S)$ acts equivariantly by automorphisms.) See Definition 13.4 for that analogous construction.

12.2. Harmonic Stretch lines between points in Teichmüller space

With these definitions in hand, we may now state our main goal for this section, a restatement of Theorem 1.6.

THEOREM 12.4. *For any two distinct hyperbolic surfaces $Y, Z \in \mathcal{T}(S)$, there is a unique harmonic stretch line proceeding from Y through Z.*

The remainder of this section is devoted to proving Theorem 12.4.

12.3. Existence of admissible triples

Let $Y, Z \in \mathcal{T}(S)$ be two hyperbolic surfaces with the optimal Lipschitz constant

$$L := \exp(d_{Th}(Y, Z)).$$

Then, for any $X \in \mathcal{T}(S)$, we have

$$E(X, Z) \leqslant L^2 E(X, Y). \tag{12.1}$$

(This is because the energy-minimizing map from X to Z will have less energy than any candidate map, in particular the composition of the energy-minimizer from X to Y with an optimal (L-)Lipschitz map from Y to Z: that composition would have energy bounded from above by $L^2 E(X, Y)$.) For any $0 \leqslant r < L^{-2}$, consider the energy difference

$$\begin{aligned} F_r \colon \mathcal{T}(S) &\longrightarrow \mathbb{R}, \\ X &\longmapsto E(X, Y) - E(X, rZ). \end{aligned}$$

By equation (12.1), we see that, for any $r \in [0, L^{-2})$,

$$F_r(X) \geqslant E(X, Y) - rL^2 E(X, Y) = (1 - rL^2) E(X, Y).$$

In particular, F_r is positive and proper. Therefore, F_r has at least one critical point. Let X_r be such a critical point. Then,

$$0 = dF_r|_{X_r} = \operatorname{Hopf}(X_r, Y) - r \operatorname{Hopf}(X_r, Z)$$

(using quadratic differentials to represent the differential of the energy functional over $\mathcal{T}(S)$ is a classic result; see, for example, [46], [92], [100], [95]). Hence,

$$\operatorname{Hopf}(X_r, Y) = r \operatorname{Hopf}(X_r, Z).$$

THEOREM 12.5. (Tholozan [89]) *For any $r \in [0, L^{-2})$, there exists a unique Riemann surface X_r in $\mathcal{T}(S)$ such that*

$$\operatorname{Hopf}(X_r, Y) = r \operatorname{Hopf}(X_r, Z).$$

Moreover, $X_r \to \infty$ as $r \to L^{-2}$.

That X_r in Theorem 12.5 diverges follows from [89, Corollary 2.3] or Lemma 6.1 in the current paper.

Using the family of harmonic map rays $\mathbf{HR}_{X_r,Y}$, we have the following existence result.

PROPOSITION 12.6. *Let Y and Z be two distinct hyperbolic surfaces in $\mathcal{T}(S)$. Then, there exists a harmonic stretch line passing through Y to Z. Consequently, there exists an admissible triple of (Y,Z).*

Proof. For any $r\in[0,L^{-2})$, let X_r be the Riemann surface obtained from Theorem 12.5. Consider the family of harmonic map rays $\mathbf{HR}_r$ which starts at X_r and passes through Y and Z. By Theorem 1.3, there exists a sequence $r_n\to L^{-2}$ such that $\mathbf{HR}_{r_n}$ converges to a Thurston geodesic, say $\mathbf{HSR}$. Then, $\mathbf{HSR}$ is a harmonic stretch line passing through Y to Z. This proves the proposition. □

12.4. Equivalent admissible triples

We begin the proof of the uniqueness part of Theorem 12.4 with the following definition.

Definition 12.7. Two admissible triples (X,f,h) and $(\widehat{X},\hat{f},\hat{h})$ of (Y,Z) are said to be *equivalent* if there exists a conformal map $\eta: X\to\widehat{X}$ such that $f=\hat{f}\circ\eta$.

It might seem unnatural that the above definition does not involve a condition on h and $\hat{h}$. However, if two admissible triples (X,f,h) and $(\widehat{X},\hat{f},\hat{h})$ of (Y,Z) are equivalent, then, because of the identity between Hopf differentials of f and h (resp, $\hat{f}$ and $\hat{h}$), we also have $h=\hat{h}\circ\eta$ by Theorem 7.1. Indeed, we see the effect of the definition as stated in the consequence described in the next lemma.

LEMMA 12.8. *Let (X,f,h) and $(\widehat{X},\hat{f},\hat{h})$ be two equivalent admissible triples of (Y,Z). Then, the harmonic stretch lines defined by them coincide.*

Proof. Let $\mathbf{HSL}: (0,\infty)\to\mathcal{T}(S)$ be the harmonic stretch line defined by (X,f,h) such that the harmonic map $f_t: X\to\mathbf{HSL}(t)\setminus\lambda$ has Hopf differential $\operatorname{Hopf}(f_t)=t\operatorname{Hopf}(f)$. In particular $\mathbf{HSL}(1)=Y$. Define $\widehat{\mathbf{HSL}}(t)$ and $\hat{f}_t$ similarly.

The assumption that (X,f,h) and $(\widehat{X},\hat{f},\hat{h})$ are equivalent implies that the composition map $\eta:=\hat{f}^{-1}\circ f$ from X to $\widehat{X}$ is conformal. Hence, $\hat{f}_t\circ\eta: X\to\widehat{\mathbf{HSL}}(t)\setminus\lambda$ is also harmonic. Moreover,

$$\begin{aligned}\operatorname{Hopf}(\hat{f}_t\circ\eta) &= \eta^*(\operatorname{Hopf}(\hat{f}_t)) = \eta^*(t\operatorname{Hopf}(\hat{f}))\\ &= t\operatorname{Hopf}(\hat{f}\circ\eta) = t\operatorname{Hopf}(f) = \operatorname{Hopf}(f_t).\end{aligned}$$

By Theorem 7.1, there is a unique crowned hyperbolic surface defined via a prescribed Hopf differential on X. In particular, $\widehat{\mathbf{HSL}}(t)=\mathbf{HSL}(t)$ holds for all t. This completes the proof. □

The following uniqueness result is key to the proof of Theorem 12.4.

PROPOSITION 12.9. *Let $(Y,Z)\in\mathcal{T}(S)\times\mathcal{T}(S)$ be a pair of distinct hyperbolic surfaces. Then, all admissible triples of (Y,Z) are equivalent.*

We prove this proposition in §12.6.

12.5. Energy difference

Note that both $f\colon X\to Y$ and $h\colon X\to L^{-2}Z$ have infinite energy. Nevertheless, we are able to talk about the energy difference of admissible triples in the following sense. Let (X,f,h) be an admissible triple of (Y,Z). Let $e(f)$ and $e(h)$ be respectively the energy densities of f and h. It then follows from Lemma 6.1 that we have the pointwise estimate $e(h)\leqslant e(f)$. Combined with Lemma 4.4, this implies that, for any compact exhaustion $\{\mathcal{K}_j\}$ of X, the limit

$$\lim_{j\to\infty} E(f|_{\mathcal{K}_j})-E(h|_{\mathcal{K}_j})$$

exists and is always a (finite) non-negative real number. Moreover, again by Lemma 4.4, the limit is independent of the choice of the compact exhaustion. Set

$$E(f)-E(h):=\lim_{j\to\infty} E(f|_{\mathcal{K}_j})-E(h|_{\mathcal{K}_j}).$$

It is clear that $E(f)-E(h)\geqslant 0$.

Definition 12.10. The *energy difference* $E(X,f,h)$ of an admissible triple (X,f,h) is defined by setting

$$E(X,f,h):=E(f)-E(h).$$

As we will see, the energy difference plays a key role in establishing the uniqueness of admissible triples.

12.6. Uniqueness of admissible triples

In this subsection, we shall prove Proposition 12.9. The idea is to show that all admissible triples of (Y,Z) have the same energy difference (Lemma 12.13) and that equality of energy difference implies equivalence of admissible triples (Lemma 12.12). The key is to compare an arbitrary admissible triple $(\widehat{X},\hat{f},\hat{h})$ to our favourite admissible triple

(X, f, h) that comes from Proposition 12.6. For Lemma 12.13, we use the continuity and properness of the energy function over $\mathcal{T}(S)$. Lemma 12.12 is more subtle. To compare the energy differences $E(f)-E(h)$ and $E(\hat{f})-E(\hat{h})$, both of which can be approximated by energy differences on Minsky regions $\mathscr{P}_R$, we first use Tholozan's local computation about energy densities to conclude that an auxiliary energy difference

$$E(f_{\mathscr{P}_R})-E(\hat{h}\circ\hat{f}^{-1}\circ f|_{\mathscr{P}_R})$$

is at least

$$E(\hat{f}|_{\mathscr{P}_R})-E(\hat{h}|_{\mathscr{P}_R}),$$

with equality holding if and only if $\hat{f}^{-1}\circ f$ is conformal. We then combine Minsky's estimate outside $\mathscr{P}_R$ and the assumption that our favourite (X, f, h) arises as a limit of minimizers of energy difference between *closed surfaces* to conclude that

$$\lim_{R\to\infty}(E(\hat{h}\circ\hat{f}^{-1}\circ f|_{\mathscr{P}_R})-E(h_{\mathscr{P}_R}))\geqslant 0$$

(Lemma 12.11). All together, this proves Lemma 12.12. We now fill in the details.

We begin by asserting a comparison for maps restricted to the Minsky region for our favourite map $h: X\to Z\setminus\lambda$. In that setting, that map h has no more energy than that of a competitor defined on a different domain, under a (naturally defined) map that changes domains.

LEMMA 12.11. *Let (X, f, h) be an admissible triple of (Y, Z) obtained from Proposition 12.6, and $(\widehat{X}, \hat{f}, \hat{h})$ be an arbitrary admissible triple of (Y, Z). Then,*

$$\lim_{R\to\infty}(E(\hat{h}\circ\hat{f}^{-1}\circ f|_{\mathscr{P}_R})-E(h|_{\mathscr{P}_R}))\geqslant 0,$$

where $\mathscr{P}_R$ is the Minsky polygonal region of $\mathrm{Hopf}(f)$.

Proof. We define the elements used in the construction of (X, f, h) (described in Proposition 12.6) as follows.

Let $0<r_n<L^{-2}$ be a sequence in which r_n converges to L^{-2} as $n\to\infty$. Let $X_n\in\mathcal{T}(S)$ be the unique Riemann surface satisfying (Theorem 12.5)

$$E(X_n, Y)-r_nE(X_n, Z)=\min_{X\in\mathcal{T}(S)}(E(X, Y)-r_nE(X, Z)).$$

Let $f_n: X_n\to Y$ and $h_n: X_n\to r_nZ$ be the corresponding harmonic maps. Then,

$$\mathrm{Hopf}(f_n)=\mathrm{Hopf}(h_n).$$

We rewrite the above equation as

$$E(f_n)-E(h_n)=\min_{X\in\mathcal{T}(S)}(E(X,Y)-r_nE(X,Z)). \tag{12.2}$$

Finally, we obtain (X,f,h) from (X_n,f_n,h_n) as in Proposition 12.6. Then, from Remark 12.3, we notice that $\hat{h}\circ\hat{f}^{-1}\colon Y\setminus\lambda\to L^{-2}Z\setminus\lambda$ extends to a 1-Lipschitz map from Y to $L^{-2}Z$. Let $\Phi_n=\mathrm{Hopf}(f_n)=\mathrm{Hopf}(h_n)$. Then, using the common Hopf differentials to provide for a common Minsky domain on which to integrate, we find

$$\begin{aligned}
&\int_{X_n\setminus\mathscr{P}_R(\Phi_n)}(e(\hat{h}\circ\hat{f}^{-1}\circ f_n)-e(h_n))\,dA_{\Phi_n} &(12.3)\\
&\quad=\int_{X_n\setminus\mathscr{P}_R(\Phi_n)}(e(\hat{h}\circ\hat{f}^{-1}\circ f_n)-2)\,dA_{\Phi_n}+\int_{X_n\setminus\mathscr{P}_R(\Phi_n)}(2-e(h_n))\,dA_{\Phi_n}\\
&\quad\leqslant\int_{X_n\setminus\mathscr{P}_R(\Phi_n)}(e(f_n)-2)\,dA_{\Phi_n}+\int_{X_n\setminus\mathscr{P}_R(\Phi_n)}(2-e(h_n))\,dA_{\Phi_n} &\text{(since } \hat{h}\circ\hat{f}^{-1} \text{ is 1-Lipschitz)}\\
&\quad\leqslant\int_{X_n\setminus\mathscr{P}_R(\Phi_n)}|e(f_n)-2|\,dA_{\Phi_n}+\int_{X_n\setminus\mathscr{P}_R(\Phi_n)}|2-e(h_n)|\,dA_{\Phi_n}\\
&\quad\leqslant 2C|\chi(S)|\exp(-R/2) &\text{(by Lemma 4.3).}
\end{aligned}$$

Since $h_n\colon X_n\to r_nZ$ is an (energy-minimizing) harmonic map between closed surfaces and that $\hat{h}\circ\hat{f}^{-1}\circ f_n\colon X_n\to L^{-2}Z$ and $h_n\colon X_n\to r_nZ$ are homotopic, it follows that

$$r_nL^2E(\hat{h}\circ\hat{f}^{-1}\circ f_n)\geqslant E(h_n).$$

The assumption that $0<r_n<L^{-2}$ then implies that

$$E(\hat{h}\circ\hat{f}^{-1}\circ f_n)\geqslant E(h_n).$$

Therefore, using that $h_n\colon X_n\to r_nZ$ minimizes energy between the *closed* surfaces X_n and r_nZ, we find

$$\int_{X_n}(e(\hat{h}\circ\hat{f}^{-1}\circ f_n)-e(h_n))\,dA_{\Phi_n}=E(\hat{h}\circ\hat{f}^{-1}\circ f_n)-E(h_n)\geqslant 0.$$

Combined with (12.3), this implies that

$$\int_{\mathscr{P}_R(\Phi_n)}(e(\hat{h}\circ\hat{f}^{-1}\circ f_n)-e(h_n))\,dA_{\Phi_n}\geqslant -2C|\chi(S)|\exp(-R/2).$$

Letting $n\to\infty$ gives

$$\int_{\mathscr{P}_R(\Phi)}(e(\hat{h}\circ\hat{f}^{-1}\circ f)-e(h))\,dA_{\Phi}\geqslant -2C|\chi(S)|\exp(-R/2),$$

where Φ is the Hopf differential of both f and h. Hence,

$$\begin{aligned}\lim_{R\to\infty} E(\hat{h}\circ\hat{f}^{-1}\circ f|_{\mathscr{P}_R(\Phi)})-E(h|_{\mathscr{P}_R(\Phi)}) &= \lim_{R\to\infty}\int_{\mathscr{P}_R(\Phi)} (e(\hat{h}\circ\hat{f}^{-1}\circ f)-e(h))\,dA_\Phi \\ &\geqslant -\lim_{R\to\infty} 2C|\chi(S)|\exp(-R/2)=0,\end{aligned}$$

which completes the proof. □

We next observe that the energy difference for a solution to the problem of Proposition 12.6 will have at least the energy difference as a solution to a similar problem with a less stringent constraint, i.e. that does not require the approximating lines to pass through exactly Y and Z.

LEMMA 12.12. *Let (X,f,h) be an admissible triple of (Y,Z) obtained from Proposition 12.6, and $(\widehat{X},\hat{f},\hat{h})$ be an arbitrary admissible triple of (Y,Z). Then,*

$$E(f)-E(h)\geqslant E(\hat{f})-E(\hat{h}),$$

where the equality holds if and only if $\hat{f}^{-1}\circ f: X\to\widehat{X}$ is conformal.

Proof. By the definition of admissible triples, we have

$$\mathrm{Hopf}(f)=\mathrm{Hopf}(h) \quad \text{and} \quad \mathrm{Hopf}(\hat{f})=\mathrm{Hopf}(\hat{h}). \tag{12.4}$$

Let $\sigma|dz|^2$ be a conformal metric on $\widehat{X}$, let $\eta:=\hat{f}^{-1}\circ f: X\to\widehat{X}$, let $\mathbf{e}:=e(\eta^{-1})$, and let $\Psi:=\mathrm{Hopf}(\eta^{-1})$ (not necessary holomorphic). Now, applying Tholozan's argument ([89, Lemma 2.5]) to the harmonic maps

$$f: X\longrightarrow Y, \quad h: X\longrightarrow L^{-2}Z,$$

and

$$\hat{f}: \widehat{X}\longrightarrow Y, \quad \hat{h}: \widehat{X}\longrightarrow L^{-2}Z,$$

we see that, for any compact subset $\widehat{K}$ of $\widehat{X}$,

$$\begin{aligned} E(f|_{\eta^{-1}(\widehat{K})})-E(\hat{h}\circ\eta|_{\eta^{-1}(\widehat{K})}) &= \int_{\widehat{K}} \frac{1}{\sqrt{1-4|\Psi|^2/\sigma^2\mathbf{e}^2}}(e(\hat{f}|_{\widehat{K}})-e(\hat{h}|_{\widehat{K}}))\,d\sigma \\ &\geqslant E(\hat{f}|_{\widehat{K}})-E(\hat{h}|_{\widehat{K}}), \end{aligned} \tag{12.5}$$

where the equality holds if and only if $\Psi\equiv 0$, i.e. η is conformal.

Let $\mathscr{P}_R$ be the Minsky's polygonal region of the Hopf differential of f. Then,

$$\begin{aligned}
&E(f)-E(h)\\
&\quad=\lim_{R\to\infty}(E(f|_{\mathscr{P}_R})-E(h|_{\mathscr{P}_R})) && \text{(by definition)}\\
&\quad=\lim_{R\to\infty}(E(f|_{\mathscr{P}_R})-E(\hat{h}\circ\eta|_{\mathscr{P}_R}))\\
&\qquad\quad+\lim_{R\to\infty}(E(\hat{h}\circ\eta|_{\mathscr{P}_R})-E(h|_{\mathscr{P}_R}))\\
&\quad\geqslant\lim_{R\to\infty}(E(f|_{\mathscr{P}_R})-E(\hat{h}\circ\eta|_{\mathscr{P}_R})) && \text{(by Lemma 12.11)}\\
&\quad\geqslant\lim_{R\to\infty}(E(\hat{f}|_{\eta(\mathscr{P}_R)})-E(\hat{h}|_{\eta(\mathscr{P}_R)})) && \text{(by (12.5))}\\
&\quad=E(\hat{f})-E(\hat{h}) && \text{(by definition),}
\end{aligned}$$

where the equality holds if and only if η is conformal. □

Next, we use the energy difference minimization property of those admissible triples from Proposition 12.6 to show that all admissible triples have the same energy difference.

LEMMA 12.13. *Let (X,f,h) be an admissible triple of (Y,Z) from Proposition 12.6, and $(\widehat{X},\hat{f},\hat{h})$ be an arbitrary admissible triple of (Y,Z). Then,*

$$E(\hat{f})-E(\hat{h})=E(f)-E(h). \tag{12.6}$$

Proof. By definitions of admissible triples and harmonic stretch lines, there exists a sequence of harmonic map rays $\mathbf{HR}_n$ converging to the (piecewise) harmonic stretch line determined by $(\widehat{X},\hat{f},\hat{h})$. Let $X_n\in\mathcal{T}(S)$ be the initial point of $\mathbf{HR}_n$.

Let $0<r_m<L^{-2}$ be a sequence which converges to L^{-2} as $m\to\infty$ and such that $f_{r_m}\colon X_{r_m}\to Y$ and $h_{r_m}\colon X_{r_m}\to r_m Z$ converge to f and h, respectively, where X_{r_m} is the Riemann surface obtained from Theorem 12.5. By Lemma 6.1, $h_{r_m}\circ(f_{r_m})^{-1}\colon Y\to r_m Z$ is 1-Lipschitz. Combined with Lemma 4.3, this implies that

$$\lim_{m\to\infty}(E(X_{r_m},Y)-r_m E(X_{r_m},Z))=E(f)-E(h).$$

Let $\varepsilon>0$ be an arbitrary positive real number. Since

$$E(X_{r_m},Y)-r_m E(X_{r_m},Z)=\min_{X\in\mathcal{T}(S)}(E(X,Y)-r_m E(X,Z)),$$

we may assume that, up to a subsequence,

$$\Big|\min_{X\in\mathcal{T}(S)}(E(X,Y)-r_m E(X,Z))-(E(f)-E(h))\Big|<\varepsilon. \tag{12.7}$$

Notice that, for any fixed $0<r<L^{-2}$,

$$\lim_{\substack{Y'\to Y\\ Z'\to Z}} \min_{X\in\mathcal{T}(S)} (E(X,Y')-rE(X,Z')) = \min_{X\in\mathcal{T}(S)} (E(X,Y)-rE(X,Z)).$$

For each m, choose X_{n_m}, $\mathbf{HR}_{n_m}$, $Y_{n_m}\in\mathbf{HR}_{n_m}$ and $Z_{n_m}\in\mathbf{HR}_{n_m}$ such that

(a) $Y_{n_m}\to Y$ and $Z_{n_m}\to Z$ as $m\to\infty$;

(b)

$$\operatorname{Hopf}(X_{n_m},Z_{n_m}) = \frac{1}{r_m}\operatorname{Hopf}(X_{n_m},Y_{n_m}),$$

i.e. X_{n_m} realizes the minimum

$$\min_{X\in\mathcal{T}(S)} (E(X,Y_{n_m})-r_mE(X,Z_{n_m}));$$

(c) $r_m<\operatorname{Lip}(Y_{n_m},Z_{n_m})^{-2}$;

(d)

$$\Big|\min_{X\in\mathcal{T}(S)} (E(X,Y_{n_m})-r_mE(X,Z_{n_m})) - \min_{X\in\mathcal{T}(S)} (E(X,Y)-r_mE(X,Z))\Big| < \varepsilon.$$

Combining the items (c) and (d) with (12.7), we see that

$$|(E(X_{n_m},Y_{n_m})-r_mE(X_{n_m},Z_{n_m}))-(E(f)-E(h))| < 2\varepsilon, \tag{12.8}$$

where we use the assumption (b) that X_{n_m} solves the energy difference minimization problem.

Let $\hat f_{n_m}: X_{n_m}\to Y_{n_m}$ and $\hat h_{n_m}: X_{n_m}\to r_mZ_{n_m}$ be the corresponding harmonic maps. We may then rewrite equation (12.8) as

$$|(E(\hat f_{n_m})-E(\hat h_{n_m}))-(E(f)-E(h))| < 2\varepsilon. \tag{12.9}$$

Now, by Lemma 6.1, $\hat h_{n_m}\circ(\hat f_{n_m})^{-1}$ is 1-Lipschitz. Combining this with Lemma 4.3, we see that

$$\Big|\lim_{m\to\infty} (E(\hat f_{n_m})-E(\hat h_{n_m}))-(E(\hat f)-E(\hat h))\Big| \leqslant 2\varepsilon.$$

The arbitrariness of ε then implies that

$$\lim_{m\to\infty} (E(\hat f_{n_m})-E(\hat h_{n_m})) = E(\hat f)-E(\hat h).$$

Combined with (12.9), this yields

$$E(\hat f)-E(\hat h) = E(f)-E(h). \qquad \square$$

Proof of Proposition 12.9. It follows directly from Lemmas 12.13 and 12.12. □

Proof of Theorem 12.4. It now follows directly from Propositions 12.9 and 12.6. □

12.7. Continuity of harmonic stretch lines

By Theorem 12.4 we see that for any two distinct hyperbolic surfaces $Y, Z \in \mathcal{T}(S)$, there is a unique harmonic stretch line $\mathbf{HSR}_{Y,Z}$ proceeding from Y through Z. A natural question one might ask is how stretch lines depend on the prescribed points Y and Z. In this regard, we have the following continuity result.

Proposition 12.14. *Let Y and Z be two distinct points in $\mathcal{T}(S)$. Let $Y_n, Z_n \in \mathcal{T}(S)$ be such that $\lim_{n\to\infty} Y_n = Y$ and $\lim_{n\to\infty} Z_n = Z$. Then, $\mathbf{HSR}_{Y_n,Z_n}$ converges to $\mathbf{HSR}_{Y,Z}$ locally uniformly as $n \to \infty$.*

Proof. By the definition of harmonic stretch lines, we see that, for every fixed n, there exists a sequence of harmonic map rays $\mathbf{HR}_{n,m}$ which converges to $\mathbf{HSR}_{Y_n,Z_n}$ locally uniformly as $m \to \infty$. Let

$$r_n := \max\{d_{Th}(Y, Y_n), d_{Th}(Y_n, Y), d_{Th}(Z, Z_n), d_{Th}(Z_n, Z)\}.$$

By assumption, we have $\lim_{n\to\infty} r_n = 0$. Now for each n, we choose a harmonic map ray $\mathbf{HR}_{n,m_n}$ whose $2r_n$-neighbourhood contains both Y and Z. This implies that the limit ray of any convergent subsequence of $\{\mathbf{HR}_{n,m_n}\}_{n\geqslant 1}$ proceeds from Y to Z. Moreover, by Lemma 6.1 and Definition 12.1, any subsequential limit is a harmonic stretch line proceeding from Y to Z. It then follows from the uniqueness part of Theorem 12.4 that $\mathbf{HR}_{n,m_n}$ converges to $\mathbf{HSR}_{Y,Z}$ locally uniformly as $n \to \infty$. This implies that $\mathbf{HSR}_{Y_n,Z_n}$ converges to $\mathbf{HSR}_{Y,Z}$ locally uniformly as $n \to \infty$. □

12.8. A characterization of admissible triples

We end this section with a characterization of harmonic stretch lines among piecewise harmonic stretch lines, in terms of their defining harmonic maps. We say that a triple $(\widehat{X}, \hat{f}, \hat{h})$ is *quasi-admissible* if it satisfies the assumptions (i)–(iii) in the definition of admissible triples.

Proposition 12.15. *A quasi-admissible triple of (Y, Z) is admissible if and only if it has the maximal energy difference among all quasi-admissible triples of (Y, Z).*

Proof. Let (X, f, h) be an admissible triple of (Y, Z) obtained from Proposition 12.6, and $(\widehat{X}, \hat{f}, \hat{h})$ be an arbitrary admissible triple of (Y, Z). Looking at the proofs of Lemmas 12.11 and 12.12, we see that the only assumptions about $(\widehat{X}, \hat{f}, \hat{h})$ that we use are that $(\widehat{X}, \hat{f}, \hat{h})$ is a quasi-admissible triple. In particular, Lemmas 12.11 and 12.12 still hold if (X, f, h) is an admissible triple obtained from Proposition 12.6 while $(\widehat{X}, \hat{f}, \hat{h})$ is a quasi-admissible triple. □

The computation at the end of the proof of Lemma 12.12 and this final result shows that the energy difference of an admissible triple is at least the energy difference of a quasi-admissible triple, with equality only if the quasi-admissible triple is actually (or conformal to) the admissible triple. Returning to the definitions of quasi-admissible and admissible triples, in the end we see that (non-admissible) quasi-admissible triples do not arise as limits of harmonic maps (or else these maps would have the same energy difference as an admissible triple and hence be admissible, acquiring the condition (iv) that the corresponding piecewise harmonic stretch line is in fact a harmonic stretch line). It may be worth noting that a distinction between the two criteria is that an admissible triple will necessarily have identical residues of the Hopf differentials at the paired punctures at a node (where the differential has a second order pole).

Note that while the energy difference of an admissible triple is *at least as large* as that for any quasi-admissible triple, every admissible triple is equivalent to the those arising as the subsequential limit of harmonic maps to Y and Z, for which the energy difference $E(\cdot,Y)-E(\cdot,rZ)$ is a *minimum* for each choice of r and declines as r tends to L^{-2} (where L is the optimal Lipschitz constant from Y to Z).

13. An "exponential map" for Thurston's metric

The goal of this section is to consider the "visual boundary" of the Thurston metric (Theorem 1.10), and define two distinct versions of a Thurston geodesic flow.

Recall that a harmonic stretch line is a limit of a sequence of harmonic map rays. By Theorem 12.4, for any two distinct hyperbolic surfaces $Y,Z\in\mathcal{T}(S)$, there is a unique harmonic stretch line proceeding from Y to Z. Here, we extend this to the following, where a *harmonic stretch ray* is a ray contained in some harmonic stretch line with the induced orientation (Recall that a harmonic stretch line admits a canonical orientation).

THEOREM 13.1. *For any hyperbolic surface $Y\in\mathcal{T}(S)$ and any projective measured lamination $[\beta]\in\mathcal{PML}(S)$, there is a unique harmonic stretch ray from Y which converges to $[\beta]$ in the Thurston compactification.*

Convention. In the remainder of this section, to simplify the notation, we will denote the dual tree $(T_\beta, 2d)$ by T_β.

13.1. Optimal equivariant Lipschitz maps to trees

Let β be a representative of $[\beta]\in\mathcal{PMF}(S)=\mathcal{PML}(S)$. Let T_β be the dual tree of the lift to the universal cover of β. Let L be the least Lipschitz constant of equivariant (surjective) maps from the universal cover $\widetilde{Y}$ to T_β. Then,

$$L \geqslant \sup_{\mu\in\mathcal{ML}(S)} \frac{2i(\beta,\mu)}{\ell_Y(\mu)}, \tag{13.1}$$

where the convention on the metric on the (dual) tree T_β implies that lengths on the tree are measured by the intersection number $2i(\beta,\mu)$. (Later, we will see that the two quantities are actually equal.) For each $0<t<L^{-1}$, consider the energy difference function $E(\cdot,Y)-t^2E(\cdot,T_\beta)$ on $\mathcal{T}(S)$. Tholozan's argument (cf. the proof of Lemma 12.12) gives a unique minimizer $X_t\in\mathcal{T}(S)$. Moreover, for each $0<r<L^{-1}$, the vertical foliation of $\operatorname{Hopf}(X_r,Y)$ is exactly $r\beta$. Letting $r\to L^{-1}$ and using Lemma 4.6 , we obtain a convergent subsequence $f_n: X_{r_n}\to Y$ which converges to a harmonic diffeomorphism $f_\infty: X_\infty\to Y\setminus\lambda$ for some chain-recurrent lamination λ. Correspondingly, the pushforward of the vertical foliation of $\operatorname{Hopf}(f_\infty)$ via f_∞ extends to a measured foliation on Y which is exactly $L^{-1}\beta$, viewed as the limit of $r_n\beta$ as $n\to\infty$. Therefore, by Proposition 7.11, the piecewise harmonic stretch line determined by f_∞ converges to $[\beta]\in\mathcal{PML}(S)$.

Recall that in terms of the natural coordinates of $\operatorname{Hopf}(f_\infty)$ ($\operatorname{Hopf}(f_\infty)=dz^2$), the pullback of the hyperbolic metric on $Y\setminus\lambda$ via f_∞ to X_∞ is

$$2(\cosh\mathscr{G}+1)\,dx^2+2(\cosh\mathscr{G}-1)\,dy^2,$$

where $\mathscr{G}=\log(1/|\nu|)$ and ν is the Beltrami differential of f_∞ (see (3.3)). Let

$$\pi: \widetilde{X}_\infty \longrightarrow L^{-1}T_\beta$$

be the projection map along leaves of vertical foliations of $\operatorname{Hopf}(\tilde f_\infty)$. By the definition of T_β, the pullback metric of T_β on $\widetilde{X}_\infty$ via π is exactly $4dx^2$. Then, the composition map $\pi\circ(\tilde f_\infty)^{-1}: \widetilde{Y}\setminus\tilde\lambda\to L^{-1}T_\beta$ is a Lipschitz map with (pointwise) Lipschitz constant

$$\sqrt{\frac{2}{\cosh\mathscr{G}+1}}<1.$$

Moreover, by Lemma 3.5, the (pointwise) Lipschitz constant of $\pi\circ(\tilde f_\infty)^{-1}$ along horizontal leaves tends to 1 as the distance to zeros of $\operatorname{Hopf}(\tilde f_\infty)$ goes to infinity. Therefore, $\pi\circ(\tilde f_\infty)^{-1}: \widetilde{Y}\setminus\tilde\lambda\to L^{-1}T_\beta$ extends to a Lipschitz map $\widetilde{Y}\to T_\beta$ whose restriction to the geodesic lamination $\tilde\lambda$ is an affine map of factor L and which has (pointwise) Lipschitz constant strictly less than L outside $\tilde\lambda$. Since λ, the projection of $\tilde\lambda$ to Y, is chain-recurrent, there exists a sequence of multicurves μ_n whose support converges to λ in the Hausdorff topology. Consequently,

$$\lim_{n\to\infty}\frac{2i(\beta,\mu_n)}{\ell_Y(\mu_n)}=L.$$

Combining this with (13.1), we see that

$$L = \sup_{\mu\in\mathcal{ML}(S)} \frac{2i(\beta,\mu)}{\ell_Y(\mu)}.$$

Moreover, this implies that every optimal Lipschitz map from $\widetilde{Y}$ to T_β would maximally stretch $\tilde{\lambda}$. We define $\tilde{\lambda}$ to be the *maximally stretched lamination* from Y to T_β, denoted by $\Lambda(Y, T_\beta)$.

Using an argument analogous to the proof of Proposition 12.6, we see that the family of harmonic map rays $\mathbf{HR}_{X_r,Y}$ determined by the harmonic map $f_r: X_r \to Y$ contains a sequence which converges to a harmonic stretch line, as r approaches L^{-2}.

In the remainder of this section, we set

$$L := \sup_{\mu\in\mathcal{MF}(S)} \frac{2i(\beta,\mu)}{\ell_Y(\mu)}.$$

We summarize the above discussion in the first two statements of the following result, while the third statement follows (as discussed) from Proposition 7.11.

PROPOSITION 13.2. *Let $Y\in\mathcal{T}(S)$ and $\beta\in\mathcal{MF}(S)$. Let T_β be the tree dual to the lift of β to the universal cover $\widetilde{S}$. Then, the following statements hold.*

- *There exists a harmonic stretch ray $\mathbf{HSR}_{Y,\beta}$ which starts at Y and which maximally stretches along the maximally stretched lamination $\Lambda(Y, T_\beta)$.*
- *There exists an equivariant Lipschitz map $f: \widetilde{Y} \to T_\beta$ whose restriction to the maximally stretched lamination $\Lambda(Y, T_\beta)$ is an affine map of factor L and which has (pointwise) Lipschitz constant strictly less than L outside $\Lambda(Y, T)$.*
- *The ray $\mathbf{HSR}_{Y,\beta}$ converges to $[\beta]$ in Thurston's compactification.*

Remark 13.3. In [88], Tabak proved that the pushforward of $\mathrm{Hopf}(f_t: X_t \to Y)$ is a subsonic ρ-holomorphic quadratic differential on Y with $\rho: Y\times\left[0, \frac{1}{4}\right) \to \mathbb{R}$ defined by $\rho(y, r) = (1-4r)^{-1/2}$. In [82], Sibner–Sibner proved a non-linear Hodge–De Rham theorem which states that for any measured foliation $\beta\in\mathcal{MF}(S)$, there exists a threshold value $t_0>0$ such that for all $0<t<t_0$, there exists a unique subsonic ρ-holomorphic quadratic differential whose vertical foliation is $t\beta$. The discussion above gives a different proof of Sibner–Sibner's result for the very specific ρ defined above and describes explicitly L^{-1} as the threshold value of β.

13.2. Admissible triples of (Y, T_β)

Having established the maximally stretched lamination $\tilde{\lambda}$ from $\widetilde{Y}$ to T_β, we are now in a position to consider admissible triples for $\widetilde{Y}$ and T_β, in the same way as in §12.1.

Combining the construction of harmonic stretch lines and the discussion in §13.1, we see that every harmonic stretch ray which starts at Y and which converges to $[\beta]\in\mathcal{PML}(S)$ in Thurston's compactification gives an optimal equivariant Lipschitz map $f:\widetilde{Y}\to T_\beta$ whose restriction to $\tilde{\lambda}$ is an affine map of factor L and which has (pointwise) Lipschitz constant strictly less than L outside $\tilde{\lambda}$.

Definition 13.4. A triple $(X,\tilde{f},\tilde{h})$ is said to be *admissible* if the following conditions hold:

(i) X is a punctured Riemann surface (possibly disconnected);

(ii) $\tilde{f}:\widetilde{X}\to\widetilde{Y}\setminus\tilde{\lambda}$ is an equivariant harmonic diffeomorphism and $\tilde{h}:\widetilde{X}\to L^{-1}T_\beta$ is an equivariant harmonic map with meromorphic Hopf differentials satisfying

$$\operatorname{Hopf}(\tilde{f})=\operatorname{Hopf}(\tilde{h});$$

(iii) the piecewise harmonic stretch line defined by the quotient map $f:X\to Y\setminus\lambda$ of $\tilde{f}:\widetilde{X}\to\widetilde{Y}\setminus\tilde{\lambda}$ is a harmonic stretch line.

Remark 13.5. By Proposition 7.11, item (ii) implies that the harmonic stretch line defined by $f:X\to Y\setminus\lambda$ converges to $[\beta]\in\mathcal{PML}(S)$. Moreover, the composition map $\tilde{h}\circ\tilde{f}^{-1}$ extends to an equivariant 1-Lipschitz map from $\widetilde{Y}$ to T_β.

Definition 13.6. Two admissible triples $(\widetilde{X},\tilde{f},\tilde{h})$ and $(\widetilde{X}',\tilde{f}',\tilde{h}')$ of (Y,T_β) are said to be *equivalent* if there exists a conformal map $\eta:\widetilde{X}\to\widetilde{X}'$ such that $\tilde{f}=\tilde{f}'\circ\eta$.

By Proposition 13.2, there exists at least one admissible triple. Applying the same argument as in §12.6, we get the following result.

PROPOSITION 13.7. *Let $Y\in\mathcal{T}(S)$ and $\beta\in\mathcal{MF}(S)$. Let T_β be the tree dual to the lift of β to the universal cover $\widetilde{S}$. Then, all admissible triples of $(\widetilde{Y},T_\beta)$ are equivalent.*

Proof. The proof is similar to that of Proposition 12.9, with (Y,Z) replaced by $(\widetilde{Y},T_\beta)$. ☐

Proof of Theorem 13.1. The existence part follows from Proposition 13.2. The uniqueness part follows from adapting the proof of Lemma 12.8 and Proposition 13.7. ☐

Proof of Theorem 1.10. The first part follows from Theorem 13.1.

For the second part, we need to show that these harmonic stretch rays are either disjoint (away from their origin Y) or coincide, and we must also show that the union of the harmonic stretch rays covers the Teichmüller space. That the union of the harmonic stretch rays covers $\mathcal{T}(S)$ follows basically from Theorem 1.6: given a point $Z\in\mathcal{T}(S)$, we take the harmonic stretch segment from Y to Z and extend it to a proper ray in $\mathcal{T}(S)$ by scaling the Hopf differential on the corresponding domain X_∞ (cf. as found via an admissible triple). This harmonic stretch ray converges to a unique point on $\mathcal{PML}(S)$ by Proposition 7.11, and hence is a leaf of the foliation. That the harmonic stretch rays from Y either coincide or are disjoint away from Y is the content of Theorem 1.6, since if two harmonic stretch lines from Y intersected at a point $Z \in \mathcal{T}(S)$, they would coincide on the harmonic stretch segment $[Y, Z]$ from Y to Z and hence extend beyond Z identically.

Finally, the third statement, that the harmonic stretch rays terminating at a point $[\eta] \in \mathcal{PML}(s)$ also foliate if we let the initial point Y vary in $\mathcal{T}(S)$, follows easily from the disjointness and surjectivity arguments of the previous paragraph. □

13.3. "Exponential map" rays

Definition 13.8. Given $Y \in \mathcal{T}(S)$ and $[\beta] \in \mathcal{PMF}(S)$, the harmonic map ray that starts at Y and converges to $[\beta]$ in the Thurston compactification (cf. Theorem 13.1) is called an *exponential map ray*, denoted by $\mathbf{ESR}_{Y,[\beta]}$.

Remark 13.9. The standard usage of the term "exponential map" in Riemannian geometry refers to a map from the tangent space at a base point to a neighborhood of that point which solves the initial value problem for geodesics with the data from the tangent space at that base point. The result is a ray structure on that neighborhood. Here, we use the term "exponential map" in terms of that resulting ray structure, but not in the sense of a map from rays in a tangent space to arcs from a base point.

Using the same argument as in the proof of Proposition 12.14, we have the following.

PROPOSITION 13.10. *Let $Y \in \mathcal{T}(S)$ and $[\eta] \in \mathcal{PML}(S)$. Assume that $Y_n \in \mathcal{T}(S)$ and $[\eta_n] \in \mathcal{PML}(S)$ are such that*

$$\lim_{n\to\infty} Y_n = Y \quad \textit{and} \quad \lim_{n\to\infty} [\eta_n] = [\eta].$$

Then, the exponential map ray $\mathbf{ESR}_{Y_n,[\eta_n]}$ converges to the exponential map ray $\mathbf{ESR}_{Y,[\eta]}$ locally uniformly as $n\to\infty$.

As a direct consequence, we obtain the following analog of Theorem 2.1.

PROPOSITION 13.11. *Let $Y \in \mathcal{T}(S)$ and $\eta \in \mathcal{ML}(S)$. Let $Y_n \in \mathcal{T}(S)$ and $\eta_n \in \mathcal{ML}(S)$ be such that*

$$\lim_{n\to\infty} Y_n = Y \quad \textit{and} \quad \lim_{n\to\infty} \eta_n = \eta.$$

Let $\Lambda(Y_n, \eta_n)$ (resp. $\Lambda(Y, \eta)$) be the maximally stretched lamination from Y_n to T_{η_n} (resp. from Y to T_η). Then, $\Lambda(Y, T_\eta)$ contains any geodesic lamination in the limit set (in the Hausdorff topology) of $\Lambda(Y_n, T_{\eta_n})$.

Proof. Let Z_n be a point in the exponential map ray $\mathbf{ESR}_{Y_n,[\eta_n]}$ such that

$$d_{Th}(Y_n, Z_n) = 1$$

and that $\mathbf{ESR}_{Y_n,[\eta_n]}$ proceeds from Y_n to Z_n. Let Z be a point in the exponential map ray $\mathbf{ESR}_{Y,[\eta]}$ such that $d_{Th}(Y,Z)=1$ and that $\mathbf{ESR}_{Y,[\eta]}$ proceeds from Y to Z. By Proposition 13.10, we see that Z_n converges to Z as $n\to\infty$. Notice that

$$\Lambda(Y_n, T_{\eta_n}) = \Lambda(Y_n, Z_n) \quad \text{and} \quad \Lambda(Y, T_\eta) = \Lambda(Y, Z).$$

The proposition now follows from Theorem 2.1. □

13.4. A comment on the infinitesimal exponential map

We prove the following result.

PROPOSITION 13.12. *Let $v \in T_Y\mathcal{T}(S)$ be a tangent vector to Teichmüller space at $Y \in \mathcal{T}(S)$. Then, there is a harmonic stretch line tangent to v through Y.*

The reader may recall that we had referred to this result in Remark 1.11.

Proof. It is well known (see, e.g., [79] or [100]) the the total energy function

$$E(\cdot) = E(\cdot, Y): \mathcal{T}(S) \longrightarrow \mathbb{R},$$

which records the total energy $E(X,Y)$ from a surface X to the surface Y, is a proper function on the Teichmüller space $\mathcal{T}(S)$; this function has a unique critical point at $Y \in \mathcal{T}(S)$, where the Hessian $\operatorname{Hess}(E)$ is positive definite. Indeed, Tromba [91], [92, Theorem 3.1.3] shows that $\operatorname{Hess}E$ at $Y \in \mathcal{T}(S)$ is a multiple of the Weil–Petersson metric on $\mathcal{T}(S)$, say $\operatorname{Hess}E = c_0 \operatorname{Hess} d^2_{WP}$ for some (universal) $c_0 > 0$, and thus

$$E = c_0 d^2_{WP} + o(d^2_{WP})$$

in a small neighborhood of $Y \in \mathcal{T}(S)$, where d_{WP} is the Weil–Petersson metric.

We consider a small "level sphere" $S(\varepsilon)$ in $\mathcal{T}(S)$ on which the Energy $E(X,Y)$ has value $E(Y,Y)+\varepsilon$. There is a standard pairing between tangent and cotangent spaces to Teichmüller space, given by integration of Beltrami differentials and holomorphic quadratic differentials; in terms of that pairing, the Hopf differential $\operatorname{Hopf}(X,Y)$ then has kernel tangent to $S(\varepsilon)$, as $dE_{|X}$ is a multiple of the Hopf differential $\operatorname{Hopf}(X,Y)$. We focus on the harmonic map rays from such points $X \in S(\varepsilon)$ through Y.

These rays are of course tangent to (the duals of) the Hopf differentials through X, but there is another distinguished curve from X and passing through Y: this is the Weil–Petersson geodesic from X to Y. As $S(\varepsilon)$ links Y in $\mathcal{T}(s)$, the tangent vectors to these geodesics fill the unit tangent sphere $T^1_Y\mathcal{T}(S)$ at $Y\in\mathcal{T}(s)$ to the Teichmüller space $\mathcal{T}(S)$: this is because the exponential map from Y surjects onto any hypersurface linking Y. Indeed, because $c_0 d^2_{WP}$ and E agree to order $o(\varepsilon^2)$, we see that $S(\varepsilon)$ is Weil–Petersson convex for small ε, and hence the map from $S(\varepsilon)$ to $T^1_Y\mathcal{T}(S)$ which records tangent vectors to the Weil–Petersson geodesics to Y has degree one.

We wish to relate the harmonic map rays to the Weil–Petersson geodesics: here we recall [96] that, analogously to the Tromba result just described, the harmonic map rays from X are Weil–Petersson geodesics at X, so that the two rays agree in C^1 up to an error of $o(\varepsilon)$. Thus, the harmonic map rays from $X\in S(\varepsilon)$ to Y agree in C^1 with the Weil–Petersson geodesics from $X\in S(\varepsilon)$ to Y to an error of $o(\varepsilon)$. Thus, the differential of the map from the normal bundle to $S(\varepsilon)$ to $T^1_Y\mathcal{T}(S)$ is the identity, up to an error of $o(1)$: we conclude that the map, to $T^1_Y\mathcal{T}(S)$, of tangents to harmonic map rays from $S(\varepsilon)$ has the same degree as the map (described just above) to the tangent vectors to the Weil–Petersson geodesics from $S(\varepsilon)$ to Y. In particular, that map from normal bundle to $S(\varepsilon)$ to $T^1_Y\mathcal{T}(S)$ has degree 1.

Finally, we exhaust $\mathcal{T}(S)$ by energy level spheres. These form a continuous family, so the rays from any of these level spheres have the same degree, as a map from the normal bundle to the sphere to $T^1_Y\mathcal{T}(S)$. So, choose $v\in T^1_Y\mathcal{T}(S)$: we can find a diverging family $X_t\in\mathcal{T}(S)$ so that the harmonic map ray from X_t through Y is tangent to $v\in T^1_Y\mathcal{T}(S)$. Then, Theorem 1.3 provides for a subsequential limit of these rays which is a Thurston geodesic and, as a locally uniform limit of smooth curves, converges in C^1, and hence passes through $v\in T^1_Y\mathcal{T}(S)$. □

13.5. Two versions of geodesic flow of Thurston metric

In this subsection, we define two versions of the geodesic flow of the Thurston metric. The first version is defined using the exponential map obtained in this section. The second version is defined using Theorem 1.1.

The exponential map we obtained in this section allows us to define a Thurston geodesic flow

$$\psi_t: \mathcal{T}(S)\times\mathcal{PML}(S)\longrightarrow\mathcal{T}(S)\times\mathcal{PML}(S)$$

as follows. For each pair $(Y,[\lambda])\in\mathcal{T}(S)\times\mathcal{PML}(S)$, let $\mathbf{ESR}^{[\lambda]}_Y:[1,\infty)\to\mathcal{T}(S)$ be the harmonic stretch ray which starts at Y and converges to $[\lambda]\in\mathcal{PML}(S)$. Then, we define the flow ψ_t by setting $\psi_t(Y,[\lambda]):=(\mathbf{ESR}^{[\lambda]}_Y(e^{2t}),[\lambda])$: that this flow is well defined follows from the second paragraph in Theorem 1.10. In particular, every ψ-orbit is a harmonic stretch line. Moreover, it follows from Theorem 1.10 that every harmonic stretch line appears as a ψ-orbit.

We next define a second version of the Thurston geodesic flow

$$\phi_t: \mathcal{T}(S)\times\mathcal{PML}(S) \longrightarrow \mathcal{T}(S)\times\mathcal{PML}(S).$$

For each pair $(Y,[\lambda])\in\mathcal{T}(S)\times\mathcal{PML}(S)$, let $\mathbf{HSR}_{Y,[\lambda]}:[1,\infty)\to\mathcal{T}(S)$ be the harmonic stretch ray obtained as the limit of harmonic map rays $\mathbf{HR}_{X_t,Y}$, where X_t degenerates along the harmonic map dual ray $\mathbf{hr}_{Y,\lambda}$(Theorem 1.1). We then define the flow ϕ_t by setting $\phi_t(Y,[\lambda])=(\mathbf{HSR}_{Y,[\lambda]}(e^{2t}),[\lambda])$. In particular, every ϕ-orbit is a harmonic stretch line. However, ϕ-orbits are "rare" in the following sense. For any non-uniquely ergodic lamination λ, the projection to $\mathcal{T}(S)$ of the orbit of $(Y,[\lambda])$ under ϕ_t is independent of the transverse measure of λ (by Lemma 8.4 and Theorem 5.7). Moreover, by [90, Theorem 10.7], for any Y and any simple *closed* curve λ, the set $\mathscr{Z}_{Y,\lambda}$, which consists of surfaces $Z\in\mathcal{T}(S)$ such that the maximally stretched lamination from Y to Z is λ, is an open subset of $\mathcal{T}(S)$. Consequently, for any fixed $Y\in\mathcal{T}(S)$, the union of projection of the ϕ orbits of $(Y,[\lambda])$, as $[\lambda]$ varies in $\mathcal{PML}(S)$, is a proper subset of $\mathcal{T}(S)$.

Concerning the continuity of both flows, we have the following result.

PROPOSITION 13.13. *The ψ_t-flow is continuous while the ϕ_t-flow is not.*

Proof. That the ψ_t-flow is continuous follows from Proposition 13.10. To see that ϕ_t is not continuous, consider two disjoint simple closed curves α and β. Let

$$\lambda_n = \alpha + \frac{1}{n}\beta.$$

Then, $\lambda_n\to\alpha$ in $\mathcal{ML}(S)$. Notice that, in the forward direction, the orbit path $\{\phi_t(X,\alpha): t\in\mathbb{R}\}$ maximally stretches exactly along α, while the orbit path $\{\phi_t(X,\lambda_n):t\in\mathbb{R}\}_n$ maximally stretches exactly along $\alpha\cup\beta$ for every $n\geqslant 1$. By Theorem 2.1, as $n\to\infty$, the orbit paths $\{\phi_t(X,\lambda_n):t\in\mathbb{R}\}_n$ do not converge to the orbit path $\{\phi_t(X,\alpha):t\in\mathbb{R}\}$. In particular, ϕ_t is not continuous at (X,α). □

How does the earthquake flow interact with the second version ϕ_t of the Thurston geodesic flow? Do they define an action of the upper triangular subgroup of $\mathrm{SL}(2,\mathbb{R})$? From the construction of piecewise harmonic stretch lines in §7, we know that the translates of harmonic stretch lines by the earthquake flow are piecewise harmonic stretch lines. Here, we show that they are also harmonic stretch lines.

PROPOSITION 13.14. *Let $\mathbf{R}$ be a harmonic stretch line in $\mathcal{T}(S)$ which maximally stretches along a measured geodesic lamination λ. Let $\mathcal{E}_\lambda(\mathbf{R})$ be a translate of $\mathbf{R}$ by an earthquake directed by λ. Then, $\mathcal{E}_\lambda(\mathbf{R})$ is also a harmonic stretch line.*

Proof. Let $Y, Z \in \mathbf{R}$ be two hyperbolic surfaces such that $\mathbf{R}$ proceeds from Y to Z. Then, by Theorem 1.6, there exists a sequence of harmonic maps $f_n: X_n \to Y$ with $X_n \in \mathcal{T}(S)$ which converges to a harmonic map $f: X \to Y \setminus \lambda$ from a punctured surface X and that (X, f) defines $\mathbf{R}$ in the sense of Theorem 1.5. Let λ_n be the horizontal measured foliation of $\mathrm{Hopf}(f_n)$. Let X'_n be the unique Riemann surface in $\mathcal{T}(S)$ such that the Hopf differential of the harmonic map $f'_n: X'_n \to \mathcal{E}_\lambda(Y)$ is also λ_n ([100]). By Lemma 4.6, the sequence of maps $\{f'_n\}$ contains a convergent subsequence, still denoted by $\{f'_n\}$ for simplicity. Let $f': X' \to \mathcal{E}_\lambda(Y) \setminus \lambda'$ be the limit harmonic diffeomorphism. Then, since λ_n limits on both λ and λ', we have that $\lambda = \lambda'$ (as geodesic laminations). Moreover, the horizontal foliations of $\mathrm{Hopf}(f)$ and $\mathrm{Hopf}(f')$ are the same. It then follows from Theorem 5.7—using the identification $Y \setminus \lambda = \mathcal{E}_\lambda(Y) \setminus \lambda' = \mathcal{E}_\lambda(Y) \setminus \lambda$—that $X' = X$ and $f' = f$. In particular, $\mathcal{E}_\lambda(\mathbf{R})$ is a harmonic stretch line defined by f'. □

Remark 13.15. As a direct consequence, the earthquake flow and the "Thurston geodesic flow" ϕ_t are compatible, and hence define an action of the upper triangular subgroup of $\mathrm{SL}(2, \mathbb{R})$. More precisely, let

$$a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \quad \text{and} \quad h_r = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}.$$

We define

$$a_t(\lambda, Y) := \phi_t(\lambda, Y) \quad \text{and} \quad h_r(\lambda, Y) := (\lambda, \mathcal{E}_{r\lambda}(Y)).$$

14. Concluding remarks

14.1. Constructing geodesics in the Teichmüller space of hyperbolic surfaces with geodesic boundary

Let $S_{g,b}$ be an orientable surface of genus g with b boundary components. Let $\mathcal{T}(S_{g,b})$ be the Teichmüller space of hyperbolic surfaces with b geodesic boundary components. There are presently three versions of a Thurston-type metric on $\mathcal{T}(S_{g,b})$ defined as follows. The first one is the so-called *arc metric/distance* introduced by Liu–Papadopoulos–Su–Théret [54]. Let $\mathcal{C}$ be the set of isotopy classes of simple closed cuves on $S_{g,b}$ and $\mathcal{A}$ be the set of isotopy classes (rel $\partial S_{g,b}$) of (essential) simple arcs on $S_{g,b}$ with endpoints on $\partial S_{g,b}$. The *arc distance* is defined as

$$d_A(X, Y) := \log \sup_{\alpha \in \mathcal{C} \cup \mathcal{A}} \frac{\ell_Y(\alpha)}{\ell_X(\alpha)}.$$

The other two versions, introduced in [3], are defined via Lipschitz maps. For $X, Y \in \mathcal{T}(S_{g,b})$, let $\mathscr{L}(X, Y)$ be the set of Lipschitz maps from X to Y that commute with the marking up to homotopy. Let $\mathrm{Lip}(\phi)$ be the Lipschitz constant of $\phi\in\mathscr{L}(X,Y)$. Define

$$d_{L\partial}(X,Y):=\log\inf\{\mathrm{Lip}(\phi):\phi\in\mathscr{L}(X,Y) \text{ with } \phi(\partial X)\subset\partial Y\},$$
$$d_{Lh}(X,Y):=\log\inf\{\mathrm{Lip}(\phi):\phi\in\mathscr{L}(X,Y) \text{ with } \phi \text{ a homeomorphism}\}.$$

Alessandrini–Disarlo [3] showed that $d_A=d_{L\partial}$ on $\mathcal{T}(S_{g,b})$ and conjectured [3, Conjecture 1.8] that $d_A=d_{Lh}$. Using harmonic stretch lines, we verify this conjecture in the case of unpunctured surfaces.

THEOREM 14.1. *With notation as above, for any $X,Y\in\mathcal{T}(S_{g,b})$, we have $d_A=d_{Lh}$. Moreover, the optimal Lipschitz constant from X to Y is always realized by a homeomorphism.*

Proof. It is clear that $d_A\leqslant d_{Lh}$. To prove the theorem, it suffices to show that the optimal Lipschitz constant from X to Y is realized by a homeomorphsim from X to Y.

Let X^d (resp. Y^d) be the double of X (resp. Y), obtained by gluing the orientation-reversing isometric copy of X (resp. Y) to X along ∂X (resp. to Y along ∂Y). Consider the harmonic stretch line $[X^d,Y^d]$ in the Teichmüller space of the double of $S_{g,b}$. The doubling process induces an involution, denoted by ι, on both X^d and Y^d. The uniqueness of harmonic stretch segment (Theorem 12.4) then implies the Lipschitz map ϕ^d from X^d to Y^d induced by any admissible triple of (X^d,Y^d) is symmetric about this involution. Hence, ϕ^d descends to a Lipschitz homeomorphism ϕ from X to Y with the same Lipschitz constant as ϕ^d. This implies that

$$d_{Lh}(X,Y)\leqslant\log\mathrm{Lip}(\phi)=d_{Th}(X^d,Y^d).$$

On the other hand, the double of any L-Lipschitz homeomorphism from X to Y gives an L-Lipschitz homeomorphism from X^d to Y^d. Hence, using that the doubling operation provides candidate maps for the minimization problem for closed surfaces from candidates for the minimization problem for the surfaces-with-boundary X and Y, we see that

$$d_{Lh}(X,Y)\geqslant d_{Th}(X^d,Y^d)=\log\mathrm{Lip}(\phi).$$

Consequently,

$$d_{Lh}(X,Y)=d_{Th}(X^d,Y^d)=\log\mathrm{Lip}(\phi).$$

Combining with the fact

$$d_A(X,Y)=d_{Th}(X^d,Y^d),$$

proved by Liu–Papadopoulos–Su–Théret [54], we know that

$$d_A(X,Y)=d_{Lh}(X,Y)=\log\mathrm{Lip}(\phi).$$ □

14.2. Relation to orthogonal foliation introduced by Choi–Dumas–Rafi and Calderon–Farre

Let $Y\in\mathcal{T}(S)$ be a hyperbolic surface and λ a measured geodesic lamination. The harmonic stretch ray obtained in Theorem 1.1 is determined by a (possibly disconnected) punctured Riemann surface X and a harmonic diffeomorphism $f\colon X\to Y\setminus\lambda$. The push-forward of the vertical foliation of the Hopf differential $\mathrm{Hopf}(f)$ extends to a measured foliation on Y which is transverse to λ. On the other hand, there is an orthogonal measured foliation associated to λ constructed in [14], [11]. If λ is maximal, then these two measured foliations coincide. A natural question is to consider the relationship between these two types of transverse measured foliations of λ on Y for non-maximal λ.

14.3. Optimal Lipschitz map from hyperbolic surface Y to negatively curve surface Z

In this paper, we have considered the sequence of minimizers of the energy difference function $E(X,Y)-tE(X,Z)$ for constant curvature surfaces Y and Z. It is natural to reflect on how these results might generalize to the case of negatively curved surfaces Y and Z for $0<t<\mathrm{Lip}(Y,Z)^{-2}$.

Appendix A. Existence for the generalized Jenkins–Serrin problem

A principal tool in this paper was an extension of the Jenkins–Serrin theory for minimal graphs in Euclidean 3-space with asymptotic boundary values to minimal graphs over hyperbolic surfaces with boundary which took values in real trees, also with asymptotic boundary values. The uniqueness theory of such graphs was described in Theorem 5.7 and used throughout the paper. In this appendix, we prove the existence part of this general Jenkins–Serrin problem; this plays a more minor role in our arguments — only appearing in the proof of Lemma 7.4 — and since the proof is rather lengthy, we relegate it to an appendix. The proof is somewhat involved, but as it turns on the structures of the foliations of the Hopf differentials, it also provides for some further development of the more technical themes of this paper. We begin with the statement of the main result; recall Definition 5.4 of "admissible dual tree" and the subsequent construction of "admissible partial boundary map" from §5.

THEOREM A.1. *Let Y be a crowned hyperbolic surface. Let T be an admissible dual tree. Then, there exists a $\pi_1(Y)$-equivariant minimal graph in $\widetilde{Y}\times T$ with a prescribed admissible partial boundary map.*

The idea of the proof is to first construct a sequence of harmonic maps $X_n \to W$ to a fixed closed surface W consisting of copies of Y which have been glued together, with controlled horizontal foliations of the associated Hopf differentials. The domains X_n for the approximating harmonic maps are found by demanding that the Hopf differentials have approximating maximal stretch laminations. We then prove that the limit of any convergent subsequence gives an equivariant minimal graph in $\widetilde{Y} \times T$ (and in fact a unique one by Theorem 5.7). The proof will be divided into two cases:

Case I: Y has no crown ends, and the measured foliation defining T consists of half-infinite cylinders corresponding to boundary components of Y and compactly supported subfoliations.

Case II: the general case.

Here, the division is entirely for expositional reasons: the basic structure of the argument will be apparent in the technically simpler Case I. Case II extends the technique to a more complicated setting.

Now, the foliations on surfaces that arise in (lifts of) minimal surfaces over (lifts of) crowned surfaces have several qualitative types: (i) they can be closed curves, or (ii) they can be part of a half-plane adjacent to a boundary leaf of a crown, or (iii) they can be a strip of bounded width with an end tending to an ideal point in a crown or an end spiralling around a boundary curve, or (iv) the leaves can be none of these: in the latter case, the leaves may be collecting into a compact portion of the surface. Of course, the technical heart of the proof comprises checking that the limiting Hopf differential on the limiting surface X_∞ has the desired trajectory structure. One crucial estimate is a bound on the diameter of the "compact part", but one also needs to make sure that the leaves, in each end, limit in the expected way.

A.1. Case I: Geodesic boundary

Let F be the measured foliation on Y whose lift to the universal cover defines T. Suppose that F consists of a half-infinite cylinders $\{A_i\}_{1 \leqslant i \leqslant a}$ with core curves $\{\alpha_i\}_{1 \leqslant i \leqslant a}$ — which are also the boundary components of Y — and b compactly supported subfoliations $\{B_i\}_{1 \leqslant i \leqslant b}$. (In this particular model case, we may take $b=1$, but we retain the notation for the more general case.) Let β_i be the measured lamination corresponding to B_i.

Let W be a closed hyperbolic surface obtained by gluing an isometric copy Y' of Y to Y along ∂Y in a way that preserves the orientation of the two copies. (This choice of orientation is not important for this case, but will be important in Case II, so we introduce it here.) Let β' be the copy of $\beta \in \mathcal{MF}(Y)$ on Y'. Consider the measured foliation

$$\mu_n := 2n\sum_i \alpha_i+\beta+\beta' \in \mathcal{MF}(W).$$

Let $X_n\in\mathcal{T}(W)$ be the Riemann surface such that the horizontal foliation of the Hopf differential of the harmonic map $f_n: X_n\to W$ is μ_n; as usual, this is guaranteed by [100]. Equivalently, the universal cover $\widetilde{X}_n$ of X_n is the unique minimal graph in $\widetilde{W}\times T_n$, where T_n is the tree dual to $\tilde{\mu}_n$.

Next, we decompose X_n as $A_{1,n}\cup...\cup A_{k,n}\cup B_n\cup B'_n$ according to the horizontal foliation of Φ_n:=Hopf(f_n), where $A_{i,n}$ is the maximal flat cylinder corresponding to the curve α_i and B_n (resp. B'_n) is the (precompact) subsurface corresponding to β (resp. β'). In particular, the width of $A_{i,n}$ is $2n$ for all i. By Lemma 3.12, we see that, for each $A_{i,n}$ and B_n, we have

$$\begin{gathered} 2n\ell_W(\alpha_i)-C \leqslant 2\|\Phi_n|_{A_{i,n}}\| \leqslant 2n\ell_W(\alpha_i)+C, \\ \ell_W(\beta)-C \leqslant 2\|\Phi_n|_{B_n}\| \leqslant \ell_W(\beta)+C, \\ \ell_W(\beta')-C \leqslant 2\|\Phi_n|_{B'_n}\| \leqslant \ell_W(\beta')+C, \end{gathered} \tag{A.1}$$

where C is a constant depending on the topology of W.

Let $\overline{B}_n\subset X_n$ be the closure of B_n.

LEMMA A.2. *There exists a constant $D>0$ depending on W such that for any $n>1$, the diameters of $\overline{B}_n$ and $\overline{B'_n}$ are at most D with respect to the $|\Phi_n|$-metric.*

Proof. We demonstrate the proof for $\{\overline{B}_n\}$. The proof of $\{\overline{B'_n}\}$ is similar. Suppose to the contrary that there exists a subsequence of $\{\overline{B}_n\}_{n\geqslant 1}$ whose diameter goes to infinity. Without loss of generality, we may assume this subsequence is $\overline{B}_n$ itself.

The area bound (A.1) of B_n implies that the injectivity radius of every point of B_n is at most D_1 for some constant D_1 depending on W. Recall that Φ_n has at most $2|\chi(W)|$ zeros. The assumption that the diameter of $\overline{B}_n$ goes to infinity implies that there exists $p_n\in B_n$ such that

$$d_n := \operatorname{dist}(p_n, \mathbf{Sing}(\Phi_n)) \to \infty,$$

where $\mathbf{Sing}(\Phi_n)$ is the zero set of Φ_n and $\operatorname{dist}(\cdot\,,\cdot)$ represents the $|\Phi_n|$-distance. By [57, Lemma 5.1], the point p_n is contained in some smooth closed regular geodesic η_n with length $\ell(\eta_n)<2D_1$. Then, any point of η_n has distance at least d_n-2D_1 from $\mathbf{Sing}(\Phi_n)$. By [57, Lemma 5.2], we see that the (maximal) cylinder C'_n containing η_n has width at least $2d_n-4D_1$. Therefore, there exists a subcylinder $C_n\subset C'_n$, with $p_n\in C_n$, of width $2d_n-4D_1-2R$ such that every point of C_n is of distance at least R to $\mathbf{Sing}(\Phi^d_n)$. (Of course, the constant R here will refer to the constant in Minsky's region $\mathscr{P}_R$.)

Next, we claim that $C_n \subset B_n$. Otherwise, suppose to the contrary that C_n is not contained in B_n, then C_n would intersect some $A_{i,n}$ because $\bigcup_i A_{i,n}$ separates B_n and B'_n. As C_n intersects both B_n and $A_{i,n}$, it follows that C_n is not horizontal. Let $q \in A_{i,n} \cap C_n$. Then, the closed geodesic $\gamma_q \subset A_{i,n}$ which contains q (and is a core curve of $A_{i,n}$) would cross C_n, because C_n is not horizontal. This implies that the $|\Phi_n|$-length γ_q is at least the width of C_n which goes to infinity. Thus, the circumference of $A_{i,n}$ also goes to infinity. On the other hand, recall that, by the construction of the measured foliation μ_n, the width of $A_{i,n}$ is $2n$. Let $\delta_{i,n}$ be the core curve of $A_{i,n}$ which is of distance n from the boundary of $A_{i,n}$. Then, by Minsky's estimate together with the fact that $A_{i,n}$ is horizontal, we see that the $|\Phi_n|$-length of $\delta_{i,n}$ converges to $\frac{1}{2}\ell_W(\alpha_i)$ as $n \to \infty$. This implies that the circumference of $A_{i,n}$ also converges to $\frac{1}{2}\ell_W(\alpha_i)$ as $n \to \infty$, which yields a contradiction.

The extremal length of η_n on B_n satisfies

$$\mathrm{Ext}_{B_n}(\eta_n) \leqslant \frac{1}{\mathrm{Mod}(C_n)} \leqslant \frac{2D_1}{2d_n - 4D_1 - 2R}.$$

By Theorem 3.10, we see that

$$E(f_n|_{B_n}) \geqslant \frac{1}{2}\frac{\ell_W^2(\eta_n)}{\mathrm{Ext}_{B_n}(\eta_n)} \geqslant \frac{(\mathrm{Syst}(W))^2(2d_n - 4D_1 - 2R)}{4D_1},$$

which goes to infinity as $n \to \infty$. This contradicts (A.1) because

$$E(f_n|_{B_n}) \leqslant 2\|\Phi_n|_{B_{1,n}}\| + 2\pi|\chi(W)|. \qquad \square$$

Proof of Theorem A.1, Case I. Let z_n be a zero of Φ_n in B_n. By Lemma A.2, the subset B_n is contained in the Minsky polygonal region P_R of X_n for $R > D$. By Lemma 4.1, we see that the sequence of pointed flat surfaces (X_n, Φ_n, z_n) contains a subsequence which converges to pointed singular flat surface $(X_\infty, \Phi_\infty, z_\infty)$; here, the flat metric $|\Phi_\infty|$ is induced by a meromorphic quadratic differential Φ_∞ which has a pole of order 2 at the pinching locus of α_i, for each i. Moreover, by Lemma A.2, the horizontal foliation of Φ_∞ consists of a half-infinite cylinders $\{A_{i,\infty}\}_{1 \leqslant i \leqslant a}$ with core curve $\{\alpha_i\}_{1 \leqslant i \leqslant a}$, and the compactly supported measured foliation β. Namely, the horizontal foliation of Φ_∞ (on the limit surface X_∞) is equivalent to the measured foliation F, whose lift to the universal cover defines T.

Up to a subsequence, we see that $\tilde{f}_n: (\widetilde{X}_n, \tilde{p}_n) \to \widetilde{W}$ converges to a harmonic map $\tilde{f}_\infty: (\widetilde{X}_\infty, \tilde{p}) \to \widetilde{W}$. Notice that the image of a core curve of A_i under $f_\infty: X \to W$ arbitrarily closely approximates the geodesic representative of α_i on W (as the distance of the core curve from the compact part grows large). Hence, the image $\tilde{f}_\infty(\widetilde{X}_\infty)$ is exactly the lift $\widetilde{Y}$ of Y. Namely, $\widetilde{X}_\infty$ is an equivariant minimal graph in the product $\widetilde{Y} \times T$. $\square$

A.2. Case II: general case

Let F be the measured foliation on Y whose universal cover defines T by duality. Suppose that F comprises a half-infinite cylinders $\{A_i\}_{1\leqslant i\leqslant a}$ foliated by closed leaves, k half-planes $\{G_j\}_{1\leqslant j\leqslant k}$, m bi-infinite strips $\{V_j\}_{1\leqslant j\leqslant m}$, and b compactly support subfoliations $\{B_i\}_{1\leqslant i\leqslant b}$. Notice that a bi-infinite strip may spiral around some half-infinite cylinder of F that is not foliated by closed leaves (this will correspond to a second order pole, with non-real residue, of the limit meromorphic differential Φ_∞ to be constructed at the end of the proof). Let $\{A_{a+i}\}_{1\leqslant i\leqslant s}$ be the set of half-infinite cylinders of F that are not foliated by closed leaves. Correspondingly, Y contains $a+s$ geodesic boundary components α_i corresponding to half-infinite cylinders A_i, and k ideal geodesic boundary arcs γ_j corresponding to half-planes G_j, as well as m ideal geodesic arcs ξ_j corresponding to V_j, and b compactly supported measured laminations β_i corresponding to B_i.

We start with the construction of a closed hyperbolic surface which consists of four copies of Y. We first take a copy Y' of Y, and then glue it back to Y with a "shear" of amount $t\neq 0$ along every ideal geodesic arc γ_i. This yields a hyperbolic surface $\widetilde{Y}$ with $2(a+s)+k$ boundary components $\alpha_1,\alpha_1',...,\alpha_{a+s},\alpha_{a+s}'$ and $\delta_1,...,\delta_k$, where α_i' is the geodesic boundary component of the copy Y' corresponding to α_i, and where δ_i is the boundary component obtained by the shearing along the ideal geodesic arc γ_i of $Y\subset\widetilde{Y}$. Moreover, the t-shearing along each ideal geodesic yields that all of δ_i have the same length $|2t|$ (all that is important here is the length of each boundary component of $\widetilde{Y}$ is positive). We then get a closed hyperbolic surface W that is obtained by gluing an isometric copy $\widetilde{Y}'$ of $\widetilde{Y}$ to $\widetilde{Y}$ in an orientation-preserving way along each geodesic boundary component. Notice that W consists of four copies of Y. We denote these four copies by Y, Y', Y'' and Y''', in such a way that the projection map $W\to\widetilde{Y}$ sends Y and Y'' (resp. Y' and Y''') to Y (resp. Y'). Each of α_i, γ_i, ξ_i and β_i gives a copy on Y', Y'' and Y''', denoted by α_i', α_i'', α_i''', γ_i', γ_i'', γ_i''', ξ_i', ξ_i'', ξ_i''', β_i', β_i'' and β_i'''. Because of the gluing process, each of the pairs $\{\gamma_i,\gamma_i'\}$, $\{\gamma_i'',\gamma_i'''\}$, $\{\alpha_i,\alpha_i''\}$ and $\{\alpha_i',\alpha_i'''\}$ are identified on W. Let γ_i, $\widehat{\gamma}_i$, α_i and α_i' be the resulting geodesics on W. Consider the geodesic lamination μ:

$$\mu:=\bigg(\bigcup_{i=1}^{a+s}(\alpha_i\cup\alpha_i')\bigg)\cup\bigg(\bigcup_{i=1}^{k}(\gamma_i\cup\widehat{\gamma}_i)\bigg)\cup\bigg(\bigcup_{i=1}^{k}\delta_i\bigg)$$
$$\cup\bigg(\bigcup_{j=1}^{m}(\xi_j\cup\xi_j'\cup\xi_j''\cup\xi_j''')\bigg)\cup\bigg(\bigcup_{j=1}^{b}(\beta_j\cup\beta_j'\cup\beta_j''\cup\beta_j''')\bigg).$$

The sublamination

$$\bigg(\bigcup_{i=1}^{a+s}(\alpha_i\cup\alpha_i')\bigg)\cup\bigg(\bigcup_{i=1}^{k}(\gamma_i\cup\widehat{\gamma}_i)\bigg)\cup\bigg(\bigcup_{i=1}^{k}\delta_i\bigg)$$

cuts W into four components which are exactly Y, Y', Y'' and Y'''.

Now, we construct a sequence of measured laminations on W whose supports converge to that of μ. Notice that both γ_i and $\widehat{\gamma}_i$ spiral around two of $\{\delta_1, ..., \delta_k\}$, say δ_{i1} and δ_{i2}. Let ${}^{\perp}\gamma_i$ (resp. ${}^{\perp}\widehat{\gamma}_i$) be the (open) geodesic arc on $W\setminus(\delta_1\cup...\cup\delta_k)$ whose closure is orthogonal to both δ_{i1} and δ_{i2}, and which is freely homotopic to γ_i (resp. $\widehat{\gamma}_i$). (Here, in the universal cover, we have that a lift of ${}^{\perp}\gamma_i$ connects lifts of δ_{i1} and δ_{i2} while a lift of γ_i might be the common asymptotic to those lifts of δ_{i1} and δ_{i2}.) We then close up ${}^{\perp}\gamma_i$ and ${}^{\perp}\widehat{\gamma}_i$ using subarcs of δ_{i1} and δ_{i2}. Denote the resulting simple closed curve by $\gamma_{i,0}$. Applying the same operation to the pairs $\{\xi_i, \xi_i''\}$ and $\{\xi_i', \xi_i'''\}$, we get simple closed curves $\xi_{i,0}$ and $\xi_{i,0}'$. The closing up process is chosen so that the geodesic representatives of $\{\gamma_{i,0}, \xi_{j,0}, \xi_{j,0}'\}_{1\leqslant i\leqslant k, 1\leqslant j\leqslant m}$ are pairwise disjoint.

For each of δ_l, α_{a+r}, and α_{a+r}', let $T_{\delta_l}^n$, $T_{\alpha_{a+r}}^n$ and $T_{\alpha_{a+r}'}^n$ be, respectively, the n-times right or left Dehn twists about δ_l, α_{a+r}, and α_{a+r}', where the direction right or left is chosen according to the spiralling direction of $\{\gamma_i, \xi_j\}_{1\leqslant i\leqslant k, 1\leqslant j\leqslant m}$ around δ_l. Let

$$\begin{aligned}
\gamma_{i,n} &:= T_{\alpha_{a+s}'}^n\circ...\circ T_{\alpha_{a+1}'}^n\circ T_{\alpha_{a+s}}^n\circ...\circ T_{\alpha_{a+1}}^n\circ T_{\delta_k}^n\circ...\circ T_{\delta_1}^n(\gamma_{i,0}),\\
\xi_{j,n} &:= T_{\alpha_{a+s}'}^n\circ...\circ T_{\alpha_{a+1}'}^n\circ T_{\alpha_{a+s}}^n\circ...\circ T_{\alpha_{a+1}}^n\circ T_{\delta_k}^n\circ...\circ T_{\delta_1}^n(\xi_{j,0}),\\
\xi_{j,n}' &:= T_{\alpha_{a+s}'}^n\circ...\circ T_{\alpha_{a+1}'}^n\circ T_{\alpha_{a+s}}^n\circ...\circ T_{\alpha_{a+1}}^n\circ T_{\delta_k}^n\circ...\circ T_{\delta_1}^n(\xi_{j,0}').
\end{aligned}$$

Then, as $n\to\infty$, the geodesic representatives of $\gamma_{i,n}$, $\xi_{j,n}$ and $\xi_{j,n}'$ converge, respectively, to the closures of $\gamma_i\cup\widehat{\gamma}_i$, $\xi_j\cup\xi_j''$, and $\xi_j'\cup\xi_j'''$. (This is elementary hyperbolic geometry, as the lifts of the curves $\gamma_{i,n}$, $\xi_{j,n}$ and $\xi_{j,n}'$ to the hyperbolic plane have unique limits, hence the ones specified.) In particular, as $n\to\infty$, the geodesic representative of

$$\sum_{i=1}^{k}\gamma_{i,n}+\sum_{j=1}^{m}(\xi_{j,n}+\xi_{j,n}')$$

converges to

$$\Bigg(\bigcup_{1\leqslant i\leqslant s}(\alpha_{a+i}\cup\alpha_{a+i}')\Bigg)\cup\Bigg(\bigcup_{1\leqslant i\leqslant k}(\gamma_i\cup\widehat{\gamma}_i)\Bigg)\cup\Bigg(\bigcup_{1\leqslant i\leqslant k}\delta_i\Bigg)\cup\Bigg(\bigcup_{1\leqslant j\leqslant m}(\xi_j\cup\xi_j'\cup\xi_j''\cup\xi_j''')\Bigg).$$

The desired sequence of measured laminations μ_n on W is defined as

$$\mu_n := n\sum_{1\leqslant i\leqslant a}(\alpha_i+\alpha_i')+n\sum_{1\leqslant i\leqslant k}\gamma_{i,n}+\mathbf{w}_j\sum_{1\leqslant j\leqslant m}(\xi_{j,n}+\xi_{j,n}')+\sum_{1\leqslant j\leqslant b}(\beta_j+\beta_j'+\beta_j''+\beta_j'''),$$

where $\mathbf{w}_j$ is the width of the horizontal strip I_j of Φ corresponding to ξ_j. Compare the constructions of μ and μ_n. It is clear that as $n\to\infty$, the support of μ_n converges to that of μ.

Subfoliations of μ_n	α_i, α'_i	$\gamma_{i,n}$	$\xi_{j,n}, \xi'_{j,n}$	$\beta_j, \beta'_j, \beta''_j, \beta'''_j$
Subsurfaces of X_n	$A_{i,n}, A'_{i,n}$	$G_{i,n}$	$V_{j,n}, V'_{j,n}$	$B_{j,n}, B'_{j,n}, B''_{j,n}, B'''_{j,n}$

Table 2. Correspondence between subfoliations of μ_n and subsurfaces of X_n.

Let $X_n \in \mathcal{T}(W)$ be the Riemann surface such that the horizontal measured foliation of the Hopf differential Φ_n of the harmonic map $f_n: X_n \to W$ is equivalent to μ_n. (Again, this follows from [100]; cf. §3.3.) We decompose X_n according to the realization of components of μ_n by the horizontal foliation of Φ_n as in Table 2, where we list the subsurfaces in the second row corresponding to the subfoliations in the first row.

The remainder of this subsection considers the convergence of the family of harmonic maps $f_n: X_n \to W$, in a certain sense. We start with the following lemma, which is an analogue of Lemma A.2.

Lemma A.3. *There exists a constant $D>0$ depending on W such that, for any $n>1$ and any $1 \leqslant j \leqslant b$, the diameter of each of $\overline{B}_{j,n}$, $\overline{B'}_{j,n}$, $\overline{B''}_{j,n}$ and $\overline{B'''}_{j,n}$ is at most D, with respect to the $|\Phi_n|$-metric.*

Proof. We show the proof for the family $\{\overline{B}_{j,n}\}_{n\geqslant 1}$. The proofs for the other three families are exactly the same. Suppose to the contrary that there exists a subsequence of $\{\overline{B}_{j,n}\}_{n\geqslant 1}$ whose diameter goes to infinity, say $\{\overline{B}_{1,n}\}_{n\geqslant 1}$. Without loss of generality, we may assume this subsequence is $\overline{B}_{1,n}$ itself. As in the proof of Lemma A.2, there exists a sequence of flat cylinders C_n (not necessary maximal) meeting $\overline{B}_{1,n}$ such that

- the circumference of C_n is less than some constant D_1;
- the width of C_n is bigger than $2d_n - 4D_1 - 2R$;
- every point of C_n is of distance at least R from the zero set of Φ_n.

(Here, $\{d_n\}$ is a divergent positive sequence and the constant R, again, refers to the constant in Minsky's region $\mathscr{P}_R$ and is chosen sufficiently large.)

Next, we claim that $C_n \subset B_{1,n}$ for n sufficiently large. In fact, if this were not the case, the boundary of $B_{1,n}$, being a union of saddle connections, would cross C_n. Then, the intersection $\partial B_{1,n} \cap C_n$ has Φ_n-length at least the width of C_n, which is bigger than $2d_n - 4D_1 - 2R$. On the other hand, by the third property of C_n mentioned above, every point of $\partial B_{1,n} \cap C_n$ is at distance at least R from the zero set of Φ. By Minsky's estimate, the image of $\partial B_{1,n} \cap C_n$ is very close to the geodesic representative of the horizontal foliation μ_n of Φ_n on W. Hence, by (3.2),

$$\ell_W([\partial B_{1,n}]) \geqslant |\partial B_{1,n} \cap C_n|_{|\Phi_n|} \geqslant 2d_n - 4D_1 - 2R,$$

which goes to infinity as $n \to \infty$. Combining 3.9 with the first and the third properties of C_n mentioned above, we see that the image of the core curve η_n of C_n on W has length bounded from above. Since there are only finitely many such curves, we may assume, up to a subsequence, that η_n belongs to a fixed homotopy class. On the other hand, note that the image on W of the core curve η_n of C_n has length bounded from below. Hence, there exists a constant $\theta\in\left[0, \frac{1}{2}\pi\right)$ such that each closed leaf of C_n is of angle at most θ with the horizontal direction. So the image of each closed leaf of C_n is contained in the ε-neighbourhood of the geodesic representative of η_n, and the image of $\partial B_{1,n}\cap C_n$ spirals around the geodesic representative of η_n. (Recall that β_n belongs a fixed homotopy class as mentioned above.) Since the support of μ_n converges to the support of μ, this implies that the core curve η_n of C_n is contained in μ, meaning that C_n is a horizontal cylinder. But each horizontal cylinder of Φ_n is either disjoint from $B_{1,n}$ or identical to $B_{1,n}$. Therefore, $C_n\subset B_{1,n}$.

The remaining part of the proof is exactly the same as that of Lemma A.2. □

Proof of Theorem A.1, Case II. Let $z_{i,n}$ be a zero of Φ_n in $B_{i,n}$. By Lemma A.3, the compact set $B_{i,n}$ is contained in the polygonal region P_R of X_n for $R>D$. By Lemma 4.1 and [59, Theorem A.3.1], we see that the sequence of the pointed flat surfaces $(X_n, \Phi_n, z_{i,n})$ contains a subsequence which converges to a singular flat surface $(X_{i,\infty}, \Phi_{i,\infty}, z_{i,\infty})$ induced by some meromorphic quadratic differential $\Phi_{i,\infty}$. Moreover, the sequence of harmonic maps $f_n: X_n\to W$ (sub)converges to a limit harmonic map $f_{i,\infty}: X_{i,\infty}\to W$ with Hopf differential $\Phi_{i,\infty}$.

Step 1. Classification of possible contributions to $(X_{i,\infty}, \Phi_{i,\infty})$ from $A_{j,n}$, $A'_{j,n}$, $G_{j,n}$, $V_{j,n}$ and $V'_{j,n}$.

Since the height of $A_{j,n}$ goes to infinity for each j as $n\to\infty$, we have that the circumference of $A_{j,n}$ converges to $\frac{1}{2}\ell_W(\alpha_j)$, by Minsky's estimate. This means that the possible contribution of $A_{j,n}$ to $(X_{i,\infty}, \Phi_{i,\infty})$ is a half-infinite horizontal cylinder. Similarly, the possible contribution of each of $A'_{j,n}$ to $(X_{i,\infty}, \Phi_{i,\infty})$ is also a half-infinite horizontal cylinder.

Notice that, since the height of $G_{j,n}$ also goes to infinity as $n\to\infty$, the image of the core curve $\gamma_{j,n}$ under $f_n: X_n\to W$ is approximately the geodesic representative of $\gamma_{j,n}$ on W. Recall that, as $n\to\infty$, the curves $\gamma_{j,n}$ converge to the union of $\gamma_j\cup\widehat{\gamma}_j$ and the two closed curves from $\{\delta_i\}$ to which they spiral. Therefore, as $n\to\infty$, the circumference of $G_{j,n}$ for each j converges to $\frac{1}{2}\ell_W(\gamma_j)$ which is infinite. This means that the possible contribution of $G_{j,n}$ to $(X_{i,\infty}, \Phi_{i,\infty})$ is a half-plane.

As for $V_{j,n}$, by Lemma 3.12, we get

$$2\|\Phi_n|_{V_{j,n}}\|\geqslant\ell_W(\xi_{j,n})-C,$$

which diverges as $n\to\infty$. Combining this with the fact that the height of $V_{j,n}$ is always $\mathbf{w}_j$, we see that, for each i, the circumference $\ell(V_{j,n})$ of $V_{j,n}$ diverges as $n\to\infty$. This implies that the possible contribution of $V_{j,n}$ to $(X_{i,\infty}, \Phi_{i,\infty})$ is an infinite strip with height $\mathbf{w}_j$. Similarly, the possible contribution of $V'_{j,n}$ to $(X_{i,\infty}, \Phi_{i,\infty})$ is also an infinite strip of height $\mathbf{w}_j$.

Step 2. Showing that $f_{i,\infty}(X_{i,\infty})$ is disjoint from any of

$$\{\alpha_l, \alpha'_l, \gamma_j, \widehat{\gamma}_j : 1 \leqslant l \leqslant a+s, 1 \leqslant j \leqslant k\}.$$

We first show that $\alpha_l \cap f_{i,\infty}(X_{i,\infty}) = \varnothing$ for $1 \leqslant l \leqslant a$. (As a reminder, these first a geodesics come from purely horizontal cylinders — there is no spiraling of the horizontal foliation.) Let $L_{l,n}$ be the central core curve of $A_{l,n}$, namely the core curve whose distance to $\partial A_{i,n}$ is $\frac{1}{2}n$. Then, by Theorem 3.9, we see that $f_n(L_{l,n})$ converges to α_l as $n \to \infty$. Now, suppose that $\alpha_l \cap f_{i,\infty}(X_{i,\infty}) \neq \varnothing$. Let p be a point in the intersection set. Then, there exists a neighbourhood U of p with $U \subset f_{i,\infty}(X_{i,\infty})$. Since $f_n: X_n \to W$ converges to $f_{i,\infty}: X_{i,\infty} \to W$, it follows that there exist small neighbourhoods $U_n \subset X_n$, with $f_n(U_n) \subset U$ and also with U_n approximating some fixed region in $X_{i,\infty}$, and hence at a uniformly bounded distance from the zeros of Φ_n. On the other hand, the discussion above implies that there exists a sequence of points $p_n \in L_{l,n}$, whose distance to the zeros of Φ_n diverges, such that $f_n(p_n) \to p \in \alpha_l$. In particular, $f_n(p_n) \in f_n(U_n)$ but $p_n \notin U_n$. This contradicts the fact that f_n is a homeomorphism, proving that $\alpha_l \cap f_{i,\infty}(i,\infty) = \varnothing$ for $1 \leqslant l \leqslant a$. Taking similar core curves of $A'_{l,n}$ and $G_{j,n}$, we see that $\alpha'_l \cap f_{i,\infty}(i,\infty) = \varnothing$ for $1 \leqslant l \leqslant a$ and $(\gamma_j \cup \widehat{\gamma}_j) \cap f_{i,\infty}(i,\infty) = \varnothing$ for $1 \leqslant j \leqslant k$.

It remains to show that $(\alpha_{a+l} \cup \alpha'_{a+l}) \cap f_{i,\infty}(X_{i,\infty}) = \varnothing$ for $1 \leqslant l \leqslant s$. Recall that, by the way we constructed the curves $\xi_{j,n}$ and $\xi'_{j,n}$, the Hausdorff limit of $\bigcup_{j=1}^{m} (\xi_{j,n} \cup \xi'_{j,n})$ contains $\bigcup_{l=1}^{s} (\alpha_{a+l} \cup \alpha'_{a+l})$. This implies that for each α_{a+l} and α'_{a+l} with $1 \leqslant l \leqslant s$, there exists $1 \leqslant j_l \leqslant k$ such that α_{a+l} and α'_{a+l} are respectively contained in the Hausdorff limit of $\xi_{j_l,n}$ and $\xi'_{j_l,n}$, as $n \to \infty$. Consider the corresponding cylinders $V_{j_l,n}$ and $V'_{j_l,n}$. By Step 1, we know that both of their circumferences diverge to infinity, as $n \to \infty$. Now, combining Theorem 3.7, Lemma 4.1 and the fact that $V_{j_l,n}$ and $V'_{j_l,n}$ have fixed width $\mathbf{w}_{j_l}$, we see that, for any $R > R_0$, there exists $c > 0$ such that each of $V_{j_l,n}$ and $V'_{j_l,n}$ crosses the Minsky's polygonal region $\mathscr{P}_R$ at most c times. Consider the intersection $\mathscr{P}_R \cap V_{j_l,n}$ (resp. $\mathscr{P}_R \cap V'_{j_l,n}$). By Theorem 3.7, the area of $\mathscr{P}_R$ is at most CR^2, and the horizontal segments of $\partial \mathscr{P}_R$ have length at most $K_1 R$. Combining with the fact that both $V_{j_l,n}$ and $V'_{j_l,n}$ have fixed width $\mathbf{w}_{j_l}$, we see that the lengths of horizontal leaves of every component of $\mathscr{P}_R \cap V_{j_l,n}$ (resp. $\mathscr{P}_R \cap V'_{j_l,n}$) are bounded above by some constant C_1. (Here, we apply the area bound to substrips of $V_{j_l,n}$ that are contained in the interior of $\mathscr{P}_R$ and the boundary estimate to substrips of $V_{j_l,n}$ that only meet the boundary $\partial \mathscr{P}_R$; the two bounds together yield the uniform bound C_1.) Therefore, the central core curve of $V_{j_l,n}$ (resp. $V'_{j_l,n}$) has total length at least $\ell(V_{j_l,n})-cC_1$ (resp. $\ell(V'_{j_l,n})-cC_1$) outside $\mathscr{P}_R$, where $\ell(V_{j_l,n})$ and $\ell(V'_{j_l,n})$ are respectively the Φ_n-circumferences of $V_{j_l,n}$ and $V'_{j_l,n}$. This then implies that for sufficiently large n, the central core curve of $V_{j_l,n}$ (resp. $V'_{j_l,n}$) contains a subsegment of length at least $\ell(V_{j_l,n})/c-C_1$ (resp. $\ell(V'_{j_l,n})/c-C_1$) which is contained in the complement $X_n\setminus\mathscr{P}_R$ of $\mathscr{P}_R$. By Theorem 3.9, the images of these segments are contained in the ε_R-neighbourhood of (and are nearly parallel to) $\xi_{j_l,n}\cup\xi'_{j_l,n}$ which converges to $\xi_{j_l}\cup\xi'_{j_l}\cup\xi''_{j_l}\cup\xi'''_{j_l}\cup\alpha_{a+l}\cup\alpha'_{a+l}$ as $n\to\infty$: here, we have chosen R so that ε_R is sufficiently small. Since both $\ell(V_{j_l,n})$ and $\ell(V'_{j_l,n})$ diverge, it follows that $\alpha_{a+l}\cup\alpha'_{a+l}$ is contained in the Hausdorff limit of the images of these segments. Now, suppose that $(\alpha_{a+l}\cup\alpha'_{a+l})\cap f_{i,\infty}(X_{i,\infty})\neq\varnothing$. Similarly to our previous argument, let p be a point in the intersection set. Then, there exists a neighbourhood U of p with $U\subset f_{i,\infty}(X_{i,\infty})$. Since $f_n\colon X_n\to W$ converges to $f_{i,\infty}\colon X_{i,\infty}\to W$, it follows that there exist small neighbourhoods $U_n\subset X_n$ with $f_n(U_n)\subset U$, but with U_n at uniformly bounded distance from the zeros of Φ_n. On the other hand, the discussion above implies that there exists a sequence of points p_n in the union of the central core curves of $V_{j_l,n}$ and $V'_{j_l,n}$, whose distance to the zeros of Φ_n diverges, such that $f_n(p_n)\to p\in\alpha_l\cup\alpha'_l$. In particular, $f_n(p_n)\in f_n(U_n)$ but $p_n\notin U_n$. This contradicts the fact that f_n is a homeomorphism, proving that $(\alpha_{a+l}\cup\alpha'_{a+l})\cap f_{i,\infty}(X_{i,\infty})=\varnothing$ for $1\leqslant l\leqslant s$.

Similarly, we see that $f_{i,\infty}(X_{i,\infty})\cap\gamma_j=\varnothing$ and $f_{i,\infty}(X_{i,\infty})\cap\widehat{\gamma}_j=\varnothing$ for $1\leqslant j\leqslant k$.

Step 3. Showing that the image $f_{i,\infty}(X_{i,\infty})$ is exactly one of Y, Y', Y'' and Y'''.

Recall that $\Phi_{i,\infty}$ is a meromorphic differential with infinite area. Let p be a pole of $\Phi_{i,\infty}$ of order at least 2.

If p has order at least 3, then there exists a neighbourhood $U(p)$ near p such that $(X_{i,\infty},\Phi_{i,\infty})$ is realized as a union of half-planes and strips. It follows from Step 1 that every such half-plane is a limit of $\{G_{j,n}\}_{g\geqslant 1,1\leqslant j\leqslant k}$ and that every strip is a limit of $\{V_{j,n},V'_{j,n}\}_{n\geqslant 1,1\leqslant j\leqslant m}$. By Theorem 3.9, the neighbourhood $U(p)$ is mapped by $f_{i,\infty}$ to a crown end whose ideal geodesic arcs are a subset of $\{\gamma_j\}_{1\leqslant j\leqslant m}$.

If p is a second-order pole with real residue, then a neighborhood of p defines a horizontal half-infinite cylinder $C(p)$ near p. By the analysis in Step 1, we see that this cylinder is a limit of $\{A_{j,n},A'_{j,n}\}_{n\geqslant 1,1\leqslant j\leqslant a}$. By Theorem 3.9, the image $f_{i,\infty}(C(p))$ is a 1-sided neighbourhood of some curve in $\{\alpha_l,\alpha'_l\}_{1\leqslant l\leqslant a}$.

If p is a second-order pole with non-real complex residue, this provides for a non-horizontal half-infinite cylinder C near p. Let $\omega_d\subset C$ be the core curve whose distance to the compact boundary of C is d. If C is vertical, then, by Theorem 3.9, we see that the length of $f_{i,\infty}(\omega_d)\subset W$ converges to zero as $d\to\infty$. This contradicts the fact that W is a closed hyperbolic surface with the shortest closed geodesic having positive length.

We are left with the case that C is neither horizontal nor vertical. Let s be the slope of C. Then, $0<|s|<\infty$. Since C is not horizontal, there exists some horizontal infinite strip crossing it. By step 1, every such strip is the limit of $\{V_{j,n}, V'_{j,n}\}_{n\geqslant 1}$. Without loss of generality, we may assume that it is the limit of $\{V_{j,n}\}_{n\geqslant 1}$. Let $\ell(C)$ be the circumference of C. Recall that $(X_{i,\infty}, \Phi_{i,\infty}, z_{i,\infty})$ is a limit of (X_n, Φ_n, z_n). There exists a non-horizontal cylinder C_n on X_n satisfying the following properties:

- the slope s_n of C_n converges to s as $n\to\infty$;
- the circumference $\ell(C_n)$ of C_n converges to $\ell(C)$ as $n\to\infty$;
- the width w_n goes to infinity as $n\to\infty$;
- the core curve of C_n is homotopic to that of C (because f_n is homotopic to the identity);
- the cylinder $V_{j,n}$ crosses C_n for sufficiently large n (as the limit of $V_{j,n}$ crosses C).

Combining the five properties of C_n mentioned above and Theorem 3.9, we see that, for large n, the image $f_n(C_n)$ on W meets a neighbourhood of the closed geodesic on W homotopic to the core curve of C. Moreover, the image $f_n(\xi_{j,n})$ spirals around this geodesic nearly

$$\frac{w_n}{\ell(C_n)/|s_n|}$$

times (recall that $\xi_{j,n}$ is the core curve of $V_{j,n}$). As $n\to\infty$, the limit of $f_n(\xi_{j,n})$ spirals infinitely many times around the geodesic homotopic to the core curve of C. Since f_n is homotopic to the identity, the limit of $\xi_{j,n}$ also spirals infinitely many times around the geodesic homotopic to the core curve of C. On the other hand, by the construction of $\xi_{j,n}$, we know that the only curves around which $\xi_{j,n}$ spirals infinitely many times is some curve in $\{\alpha_{a+l}, \alpha'_{a+l}, \delta_j\}_{1\leqslant l\leqslant s, 1\leqslant j\leqslant k}$. Therefore, the core curve of C is homotopic to some curve in $\{\alpha_{a+l}, \alpha'_{a+l}, \delta_j\}_{1\leqslant l\leqslant s, 1\leqslant j\leqslant k}$. It then follows from Theorem 3.9 that $f_{i,\infty}(C)$ is a 1-sided neighbourhood of this curve.

Summarizing the above discussion in this step, we see that the image $f_{i,\infty}(X_{i,\infty})$ is a crowned surface bounded by some curves from $\{\alpha_l, \alpha'_l, \gamma_j, \widehat{\gamma}_j, \delta_j : 1\leqslant l\leqslant a+s, 1\leqslant j\leqslant k\}$. Combining this with Step 2, we see that $f_{i,\infty}(X_{i,\infty})$ is one of Y, Y', Y'' and Y''', the four components of the complement of

$$\biggl(\bigcup_{1\leqslant i\leqslant a+s} (\alpha_i \cup \alpha'_i)\biggr) \cup \biggl(\bigcup_{1\leqslant i\leqslant k} (\gamma_i \cup \widehat{\gamma}_i)\biggr) \cup \biggl(\bigcup_{1\leqslant i\leqslant k} \delta_i\biggr).$$

Step 4. Determining the horizontal foliation of Φ_∞.

For any $1\leqslant i\leqslant b$, by Steps 1 and 3, we see that $\Phi_{i,\infty}$ contains a compactly supported foliation B_i corresponding to the lamination β_i, and that the image of

$$f_{i,\infty} : (X_{i,\infty}, \Phi_{i,\infty}) \longrightarrow W$$

is Y, since $\beta_i \subset Y$ (and not one of the copies Y', Y'' and Y'''). Since $\beta_i \subset Y$ (and, again, not one of the copies Y', Y'' and Y''') for all $i=1,2,...,b$, we then see that we may apply a diagonal argument and choose a sequence of points

$$z_n \in \bigcup_{1\leqslant i\leqslant b} \overline{B}_{i,n} \subset X_n,$$

so that there exists a subsequence of $f_n: (X_n, \Phi_n, z_n) \to W$ with this property: the subsequence converges to a harmonic map $f_\infty: (X_\infty, \Phi_\infty, z_\infty) \to Y$ and the horizontal foliation of the Hopf differential Φ_∞ of f_∞ contains all of the laminations $\beta_1, ..., \beta_b$. By Step 1, we see that the horizontal foliation of Φ_∞ contains a half-infinite cylinders corresponding to $\{A_{i,n}\}_{1\leqslant i\leqslant a}$, k half-planes corresponding to $\{G_{j,n}\}_{1\leqslant j\leqslant k}$, and b compactly supported foliations corresponding to $\{B_{j,n}\}_{1\leqslant j\leqslant b,}$. It remains to consider the contributions from

$$\{V_{j,n}, V'_{j,n}\}_{1\leqslant j\leqslant m} \quad \text{and} \quad \{B'_{j,n}, B''_{j,n}, B'''_{j,n}\}_{1\leqslant j\leqslant b}.$$

By the construction of $\xi_{j,n}$ for each $1\leqslant j\leqslant m$, (the geodesic representative of) the image of the core curve $\xi_{j,n}$ of $V_{j,n}$ converges to the union of $\xi_j \cup \xi''_j$ and the closed geodesics to which they spiral; this becomes clear when one lifts the families to $\mathbb{H}^2$. Combined with the fact that $\xi_j \subset Y$, this means that some portion of $V_{j,n}$ survives in Φ_∞. By Step 1, the contribution from $\{V_{j,n}\}_{n\geqslant 1}$ is an infinite strip. Consequently, the horizontal foliation of Φ_∞ contains m strips corresponding to $\{V_{j,n}\}_{1\leqslant j\leqslant m}$. On the other hand, if Φ_∞ contains some contribution from $\{V'_{j,n}\}_{n\geqslant 1}$, then by Step 1 this contribution is an infinite strip whose image on W contains curves homotopic to γ'_i or γ'''_i; recall that we have identified γ'''_i and γ''_i. But neither γ'_i nor γ'''_i is homotopic to some curve contained in Y. Hence, the horizontal foliation of Φ_∞ contains no contribution from any of

$$\{V'_{j,n}\}_{1\leqslant j\leqslant m, n\geqslant 1}.$$

Finally, suppose that the horizontal foliation of Φ_∞ contains some contribution from $\{\beta'_j, \beta''_j, \beta'''_j\}_{1\leqslant j\leqslant b, n\geqslant 1}$, say β'_j. Then, by Lemma A.3, it contains the whole of β'_j. Correspondingly, $Y = f_\infty(X_\infty)$ contains $f_\infty(\beta'_j)$, which is homotopic to β'_j (on Y'). This contradicts the fact that β'_j is contained in Y' instead of Y. Therefore, the horizontal foliation of Φ_∞ does not contain any contribution from

$$\{\beta'_j, \beta''_j, \beta'''_j\}_{1\leqslant j\leqslant b, n\geqslant 1}.$$

Summarizing the discussion above, we see that the horizontal foliation of $\Phi_{i,\infty}$ consists of a half-infinite cylinders corresponding to $\{\alpha_i\}_{1\leqslant i\leqslant a}$, k half-planes corresponding to $\{\gamma_j\}_{1\leqslant j\leqslant k}$, m bi-infinite strips corresponding to $\{\xi_j\}_{1\leqslant j\leqslant m}$ and b compactly supported foliations corresponding to $\{\beta_j\}_{1\leqslant j\leqslant b}$.

Step 5. Constructing a minimal graph.

By Step 4, we know that the horizontal foliation of Φ_∞ consists of a half-infinite cylinders corresponding to $\{\alpha_i\}_{1\leqslant i\leqslant a}$, k half-planes corresponding to $\{\gamma_j\}_{1\leqslant j\leqslant k}$, m bi-infinite strips corresponding to $\{\xi_j\}_{1\leqslant j\leqslant m}$ and b compactly supported foliations corresponding to $\{\beta_j\}_{1\leqslant j\leqslant b}$. In other words, the horizontal foliation of Φ_∞ is equivalent to the measured foliation F, whose lift to the universal cover defines T. Therefore, X_∞ is a minimal graph in $\widetilde{Y}\times T$. This completes the proof. □

Remark A.4. In the above proof, we implicitly assume that the measured foliation F has non-trivial compact components. If the measured foliation F has no compact component (i.e. the foliation comprises only half-infinite cylinders, half-planes, and infinite strips), then we simply take z_n to be an arbitrary zero of Φ_n.

For the convenience of reference, we also summarize the construction in this appendix as follows.

PROPOSITION A.5. *For any crowned hyperbolic surface Y and any admissible measured foliation F on Y, there exist*

- *a closed hyperbolic surface W and a chain recurrent geodesic lamination λ on W such that $W\setminus\lambda$ is the union of two or four isometric copies of Y, and*
- *a sequence of Riemann surfaces $X_n\in\mathcal{T}(W)$ and a sequence of harmonic diffeomorphisms $f_n: X_n\to W$ homotopic to the identity which converge to a harmonic diffeomorphism $f_\infty: X_\infty\to W\setminus\lambda$ (in the sense of Definition 4.5) from some punctured Riemann surface X_∞ such that, on each component X'_∞ of X_∞, the horizontal measured foliation of the Hopf differential of $f_\infty|_{X'_\infty}: X'_\infty\to Y$ is equivalent to F.*

The ideas in this section also lead to the following characterization of Thurston stretch lines.

COROLLARY A.6. *Let $Y\in\mathcal{T}(S)$ be a hyperbolic surface and λ be a maximal geodesic lamination. Let $\mathbf{SR}_{Y,\lambda}$ be the Thurston stretch line determined by Y and λ. Then, $\mathbf{SR}_{Y,\lambda}$ is a harmonic stretch line if and only if λ is chain-recurrent.*

Proof. Suppose that $\mathbf{SR}_{Y,\lambda}$ is a harmonic stretch line. Then, by definition, there exists a sequence of harmonic map rays $\mathbf{HR}_{X_n,Y}$ for some $X_n\in\mathcal{T}(S)$ that converges to $\mathbf{SR}_{Y,\lambda}$ as $n\to\infty$.

By Lemma 4.6, the sequence of harmonic maps $f_n: X_n\to Y$ (sub)converges to a harmonic diffeomorphism $f_\infty: X_\infty\to Y\setminus\mu$ for some chain-recurrent lamination $\mu\subset\lambda'$. Since by assumption $\mathbf{SR}_{Y,\lambda}$ is the limit of $\mathbf{HR}_{X_n,Y}$, we see by Proposition 8.1 (ii) that $\mathbf{SR}_{Y,\lambda}$ is the piecewise harmonic stretch line constructed from $f_\infty: X_\infty\to Y\setminus\mu$ in Theorem 1.5.

In particular, the harmonic stretch line $\mathbf{SR}_{Y,\lambda}$ maximally stretches exactly μ. Therefore, $\mu{=}\lambda$. This implies that λ is chain-recurrent.

Now, we turn to the other direction. Suppose that λ is chain-recurrent. Then, there exists a sequence of multicurves α_n which converges to λ in the Hausdorff topology (on the set of geodesic laminations on Y), as $n\to\infty$. Let X_n be the Riemann surface such that the horizontal foliation of the Hopf differential of the harmonic map $X_n\to Y$ is $n\alpha_n$. Let $\mathbf{HR}_{X_n,Y}$ be the corresponding harmonic map ray. By Lemma 4.6, we see that $\{\mathbf{HR}_{X_n,Y}\}_{n\geqslant 1}$ contains a subsequence which converges to some harmonic stretch line **HSR**. Applying the idea of the proof of Theorem A.1, we conclude that the limiting harmonic stretch line **HSR** maximally stretches along λ. The assumption that λ is maximal then implies that $\mathbf{HSR}{=}\mathbf{SR}_{Y,\lambda}$ (cf. Lemma 7.10). □

References

[1] Ahlfors, L. V., Some remarks on Teichmüller's space of Riemann surfaces. *Ann. of Math.*, 74 (1961), 171–191.

[2] Al′ber, S. I., On n-dimensional problems in the calculus of variations in the large. *Sov. Math. Dokl.*, 5 (1964), 700–704.

[3] Alessandrini, D. & Disarlo, V., Generalizing stretch lines for surfaces with boundary. *Int. Math. Res. Not. IMRN*, (2022), 18919–18991.

[4] Berger, M. S., *Nonlinearity and Functional Analysis.* Lectures on Nonlinear Problems in Mathematical Analysis. Pure Appl. Math. Academic Press, New York–London, 1977.

[5] Bers, L., Quasiconformal mappings and Teichmüller's theorem, in *Analytic Functions*, pp. 89–119. Princeton Univ. Press, Princeton, NJ, 1960.

[6] Bonahon, F., Earthquakes on Riemann surfaces and on measured geodesic laminations. *Trans. Amer. Math. Soc.*, 330 (1992), 69–95.

[7] — Shearing hyperbolic surfaces, bending pleated surfaces and Thurston's symplectic form. *Ann. Fac. Sci. Toulouse Math.*, 5 (1996), 233–297.

[8] — Geodesic laminations on surfaces, in *Laminations and Foliations in Dynamics, Geometry and Topology* (Stony Brook, NY, 1998), Contemp. Math., 269, pp. 1–37. Amer. Math. Soc., Providence, RI, 2001.

[9] Bonsante, F., Mondello, G. & Schlenker, J. M., A cyclic extension of the earthquake flow I. *Geom. Topol.*, 17 (2013), 157–234.

[10] — A cyclic extension of the earthquake flow II. *Ann. Sci. Éc. Norm. Supér.*, 48 (2015), 811–859.

[11] Calderon, A. & Farre, J., Shear-shape cocycles for measured laminations and ergodic theory of the earthquake flow. *Geom. Topol.*, 28 (2024), 1995–2124.

[12] Cheng, S. Y., Liouville theorem for harmonic maps, in *Geometry of the Laplace Operator* (Honolulu, 1979), Proc. Sympos. Pure Math., XXXVI, pp. 147–151. Amer. Math. Soc., Providence, RI, 1980.

[13] Cheng, S. Y. & Yau, S.-T., Differential equations on Riemannian manifolds and their geometric applications. *Comm. Pure Appl. Math.*, 28 (1975), 333–354.

[14] Choi, Y.-E., Dumas, D. & Rafi, K., Grafting rays fellow travel Teichmüller geodesics. *Int. Math. Res. Not. IMRN*, 2012 (2012), 2445–2492.

[15] Choi, Y.-E. & Rafi, K., Comparison between Teichmüller and Lipschitz metrics. *J. Lond. Math. Soc.*, 76 (2007), 739–756.
[16] Choi, Y.-E., Rafi, K. & Series, C., Lines of minima and Teichmüller geodesics. *Geom. Funct. Anal.*, 18 (2008), 698–754.
[17] — Lines of minima are uniformly quasigeodesic. *Pacific J. Math.*, 237 (2008), 21–44.
[18] Colding, T., *Introduction to Partial Differential Equations*. Lecture Two: The Gradient Estimate. MIT OpenCourseWare. Available at https://ocw.mit.edu/courses/18-152-introduction-to-partial-differential-equations-fall-2005/resources/lecture2.
[19] Collin, P. & Rosenberg, H., Construction of harmonic diffeomorphisms and minimal graphs. *Ann. of Math.*, 172 (2010), 1879–1906.
[20] Daskalopoulos, G. & Uhlenbeck, K., Analytic properties of Stretch maps and geodesic laminations. Preprint, 2022. arXiv:2205.08250[math.DG].
[21] — Transverse measures and best Lipschitz and least gradient maps. *J. Differential Geom.*, 127 (2024), 969–1018.
[22] Dumas, D., Lenzhen, A., Rafi, K. & Tao, J., Coarse and fine geometry of the Thurston metric. *Forum Math. Sigma*, 8 (2020), Paper No. e28, 58 pp.
[23] Eells, J. & Lemaire, L., Deformations of metrics and associated harmonic maps. *Proc. Indian Acad. Sci. Math. Sci.*, 90 (1981), 33–45.
[24] Eells, J., Jr. & Sampson, J. H., Harmonic mappings of Riemannian manifolds. *Amer. J. Math.*, 86 (1964), 109–160.
[25] Farb, B. & Wolf, M., Harmonic splittings of surfaces. *Topology*, 40 (2001), 1395–1414.
[26] Fathi, A., Laudenbach, F. & Poénaru, V., *Thurston's Work on Surfaces*. Math. Notes, 48. Princeton Univ. Press, Princeton, NJ, 2012.
[27] Gardiner, F. P., *Teichmüller Theory and Quadratic Differentials*. John Wiley & Sons, New York, 1987.
[28] Gardiner, F. P. & Masur, H., Extremal length geometry of Teichmüller space. *Complex Variables Theory Appl.*, 16 (1991), 209–237.
[29] Gerstenhaber, M. & Rauch, H. E., On extremal quasi-conformal mappings. I. *Proc. Nat. Acad. Sci. U.S.A.*, 40 (1954), 808–812.
[30] — On extremal quasi-conformal mappings. II. *Proc. Nat. Acad. Sci. U.S.A.*, 40 (1954), 991–994.
[31] Gupta, S., Asymptoticity of grafting and Teichmüller rays. *Geom. Topol.*, 18 (2014), 2127–2188.
[32] — Asymptoticity of grafting and Teichmüller rays II. *Geom. Dedicata*, 176 (2015), 185–213.
[33] — Limits of harmonic maps and crowned hyperbolic surfaces. *Trans. Amer. Math. Soc.*, 372 (2019), 7573–7596.
[34] — Harmonic maps and wild Teichmüller spaces. *J. Topol. Anal.*, 13 (2021), 349–393.
[35] Gupta, S. & Wolf, M., Quadratic differentials, half-plane structures, and harmonic maps to trees. *Comment. Math. Helv*, 91 (2016), 317–356.
[36] Hamilton, R. S., *Harmonic Maps of Manifolds with Boundary*. Lecture Notes in Math., 471. Springer, Heidelberg–New York, 1975.
[37] Hartman, P., On homotopic harmonic maps. *Canadian J. Math.*, 19 (1967), 673–687.
[38] Heinz, E., On certain nonlinear elliptic differential equations and univalent mappings. *J. Analyse Math.*, 5 (1956/57), 197–272.
[39] Hitchin, N. J., The self-duality equations on a Riemann surface. *Proc. London Math. Soc.*, 55 (1987), 59–126.

[40] Huang, Y., Ohshika, K. & Papadopoulos, A., The infinitesimal and global Thurston geometry of Teichmüller space. *J. Differential Geom.*, 131 (2025), 311–400.
[41] Huang, Y. & Papadopoulos, A., Optimal Lipschitz maps on one-holed tori and the Thurston metric theory of Teichmüller space. *Geom. Dedicata*, 214 (2021), 465–488.
[42] Hubbard, J. & Masur, H., Quadratic differentials and foliations. *Acta Math.*, 142 (1979), 221–274.
[43] Imayoshi, Y. & Taniguchi, M., *An Introduction to Teichmüller Spaces*. Springer, Tokyo, 1992.
[44] Jenkins, H. & Serrin, J., Variational problems of minimal surface type. II. Boundary value problems for the minimal surface equation. *Arch. Rational Mech. Anal.*, 21 (1966), 321–342.
[45] Jost, J., *Harmonic Maps between Surfaces*. Lecture Notes in Math., 1062. Springer, Heidelberg, 1984.
[46] — *Two-Dimensional Geometric Variational Problems*. Pure Appl. Math. John Wiley & Sons, Chichester, 1991.
[47] Kerckhoff, S. P., The asymptotic geometry of Teichmüller space. *Topology*, 19 (1980), 23–41.
[48] — The Nielsen realization problem. *Ann. of Math.*, 117 (1983), 235–265.
[49] — Earthquakes are analytic. *Comment. Math. Helv.*, 60 (1985), 17–30.
[50] — Lines of minima in Teichmüller space. *Duke Math. J.*, 65 (1992), 187–213.
[51] Korevaar, N. J. & Schoen, R. M., Sobolev spaces and harmonic maps for metric space targets. *Comm. Anal. Geom.*, 1 (1993), 561–659.
[52] Lenzhen, A., Rafi, K. & Tao, J., Bounded combinatorics and the Lipschitz metric on Teichmüller space. *Geom. Dedicata*, 159 (2012), 353–371.
[53] Levitt, G., Foliations and laminations on hyperbolic surfaces. *Topology*, 22 (1983), 119–135.
[54] Liu, L., Papadopoulos, A., Su, W. & Théret, G., On length spectrum metrics and weak metrics on Teichmüller spaces of surfaces with boundary. *Ann. Acad. Sci. Fenn. Math.*, 35 (2010), 255–274.
[55] Marković, V., Uniqueness of minimal diffeomorphisms between surfaces. *Bull. Lond. Math. Soc.*, 53 (2021), 1196–1204.
[56] Maskit, B., Comparison of hyperbolic and extremal lengths. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 10 (1985), 381–386.
[57] Masur, H. & Smillie, J., Hausdorff dimension of sets of nonergodic measured foliations. *Ann. of Math.*, 134 (1991), 455–543.
[58] Mazón, J. M., Rossi, J. D. & Segura de León, S., Functions of least gradient and 1-harmonic functions. *Indiana Univ. Math. J.*, 63 (2014), 1067–1084.
[59] McMullen, C., Amenability, Poincaré series and quasiconformal maps. *Invent. Math.*, 97 (1989), 95–127.
[60] Mese, C., A variational construction of the Teichmüller map. *Calc. Var. Partial Differential Equations*, 21 (2004), 15–46.
[61] Minsky, Y. N., Harmonic maps, length, and energy in Teichmüller space. *J. Differential Geom.*, 35 (1992), 151–217.
[62] Mirzakhani, M., Ergodic theory of the earthquake flow. *Int. Math. Res. Not. IMRN*, (2008), Art. ID rnm116, 39 pp.
[63] Morgan, J. W. & Otal, J.-P., Relative growth rates of closed geodesics on a surface under varying hyperbolic structures. *Comment. Math. Helv.*, 68 (1993), 171–208.
[64] Mulase, M. & Penkava, M., Ribbon graphs, quadratic differentials on Riemann surfaces, and algebraic curves defined over $\overline{\mathbf{Q}}$. *Asian J. Math.*, 2 (1998), 875–919.

[65] Mumford, D., A remark on Mahler's compactness theorem. *Proc. Amer. Math. Soc.*, 28 (1971), 289–294.

[66] Nelli, B. & Rosenberg, H., Minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$. *Bull. Braz. Math. Soc.*, 33 (2002), 263–292. Correction in *Bull. Braz. Math. Soc.*, 38 (2007), 661–664.

[67] Okumura, Y., Global real analytic length parameters for Teichmüller spaces. *Hiroshima Math. J.*, 26 (1996), 165–179.

[68] Pan, H., Local rigidity of the Teichmüller space with the Thurston metric. *Sci. China Math.*, 66 (2023), 1751–1766.

[69] Pan, H. & Su, W., The geometry of the Thurston metric: A survey, in *In the Tradition of Thurston* III, pp. 7–43. Springer, Cham, 2024.

[70] Pan, H. & Wolf, M., Envelopes of the Thurston metric on Teichmüller space. Preprint, 2024. arXiv:2401.06607[math.GT].

[71] Papadopoulos, A., Ideal triangles, hyperbolic surfaces and the Thurston metric on Teichmüller space, in *Moduli Spaces and Locally Symmetric Spaces*, pp. 135–181. Int. Press, Somerville, MA; Higher Education Press, Beijing, 2021.

[72] Papadopoulos, A. & Théret, G., On Teichmüller's metric and Thurston's asymmetric metric on Teichmüller space, in *Handbook of Teichmüller Theory*. Vol. I, IRMA Lect. Math. Theor. Phys., 11, pp. 111–204. Eur. Math. Soc., Zürich, 2007.

[73] Papadopoulos, A. & Yamada, S., Deforming hyperbolic hexagons with applications to the arc and the Thurston metrics on Teichmüller spaces. *Monatsh. Math.*, 182 (2017), 913–939.

[74] Penner, R. C. & Harer, J. L., *Combinatorics of Train Tracks*. Ann. of Math. Stud., 125. Princeton Univ. Press, Princeton, NJ, 1992.

[75] Sampson, J. H., Some properties and applications of harmonic mappings. *Ann. Sci. Éc. Norm. Supér.*, 11 (1978), 211–228.

[76] Scherk, H. F., Bemerkungen über die kleinste Fläche innerhalb gegebener Grenzen. *J. Reine Angew. Math.*, 13 (1835), 185–208.

[77] Schmutz, P., Die parametrisierung des Teichmüllerraumes durch geodätische Längenfunktionen. *Comment. Math. Helv.*, 68 (1993), 278–288.

[78] Schoen, R. & Yau, S.-T., On univalent harmonic maps between surfaces. *Invent. Math.*, 44 (1978), 265–278.

[79] — Existence of incompressible minimal surfaces and the topology of three-dimensional manifolds with nonnegative scalar curvature. *Ann. of Math.*, 110 (1979), 127–142.

[80] Schoen, R. M., Analytic aspects of the harmonic map problem, in *Seminar on Nonlinear Partial Differential Equations* (Berkeley, CA, 1983), Math. Sci. Res. Inst. Publ., 2, pp. 321–358. Springer, New York, 1984.

[81] — The role of harmonic mappings in rigidity and deformation problems, in *Complex Geometry* (Osaka, 1990), Lect. Notes Pure and Appl. Math., 143, pp. 179–200. Dekker, New York, 1993.

[82] Sibner, L. M. & Sibner, R. J., A non-linear Hodge–de Rham theorem. *Acta Mathematica*, 125 (1970), 57–73.

[83] Slegers, I., Equivariant harmonic maps depend real analytically on the representation. *Manuscripta Math.*, 169 (2022), 633–648.

[84] Sternberg, P., Williams, G. & Ziemer, W. P., Existence, uniqueness, and regularity for functions of least gradient. *J. Reine Angew. Math.*, 430 (1992), 35–60.

[85] Sternberg, P. & Ziemer, W. P., The Dirichlet problem for functions of least gradient, in *Degenerate Diffusions* (Minneapolis, MN, 1991), IMA Vol. Math. Appl., 47, pp. 197–214. Springer, New York, 1993.

[86] Strebel, K., *Quadratic Differentials.* Ergebnisse der Mathematik und ihrer Grenzgebiete, 5. Springer, Heidelberg, 1984.

[87] Su, W., Problems on the Thurston metric, in *Handbook of Teichmüller Theory.* Vol. V, IRMA Lect. Math. Theor. Phys., 26, pp. 55–72. Eur. Math. Soc., Zürich, 2016.

[88] Tabak, B., A geometric characterization of harmonic diffeomorphisms between surfaces. *Math. Ann.*, 270 (1985), 147–157.

[89] Tholozan, N., Dominating surface group representations and deforming closed anti-de Sitter 3-manifolds. *Geom. Topol.*, 21 (2017), 193–214.

[90] Thurston, W. P., Minimal stretch maps between hyperbolic surfaces, in *Collected Works of William P. Thurston with Commentary.* Vol. I. Foliations, Surfaces and Differential Geometry, pp. 533–585. Amer. Math. Soc., Providence, RI, 2022. 1986 preprint, 1998 eprint.

[91] Tromba, A. J., On an energy function for the Weil–Petersson metric on Teichmüller space. *Manuscripta Math.*, 59 (1987), 249–260.

[92] — *Teichmüller Theory in Riemannian Geometry.* Lectures in Mathematics ETH Zürich. Birkhäuser, Basel, 1992.

[93] Walsh, C., The horoboundary and isometry group of Thurston's Lipschitz metric, in *Handbook of Teichmüller Theory.* Vol. IV, IRMA Lect. Math. Theor. Phys., 19, pp. 327–353. Eur. Math. Soc., Zürich, 2014.

[94] Wan, T. Y.-H., Constant mean curvature surface, harmonic maps, and universal Teichmüller space. *J. Differential Geom.*, 35 (1992), 643–657.

[95] Wentworth, R. A., Energy of harmonic maps and Gardiner's formula, in *In the Tradition of Ahlfors–Bers.* IV, Contemp. Math., 432, pp. 221–229. Amer. Math. Soc., Providence, RI, 2007.

[96] Wolf, M., The Teichmüller theory of harmonic maps. *J. Differential Geom.*, 29 (1989), 449–479.

[97] — High energy degeneration of harmonic maps between surfaces and rays in Teichmüller space. *Topology*, 30 (1991), 517–540.

[98] — Infinite energy harmonic maps and degeneration of hyperbolic surfaces in moduli space. *J. Differential Geom.*, 33 (1991), 487–539.

[99] — On realizing measured foliations via quadratic differentials of harmonic maps to $\mathbb{R}$-trees. *J. Anal. Math.*, 68 (1996), 107–120.

[100] — Measured foliations and harmonic maps of surfaces. *J. Differential Geom.*, 49 (1998), 437–467.

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