LIMITS OF CUBIC DIFFERENTIALS AND BUILDINGS

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Abstract. In the Labourie-Loftin parametrization of the Hitchin component of surface group representations into $\mathrm{SL}(3,\mathbb{R})$, we prove an asymptotic formula for holonomy along rays in terms of local invariants of the holomorphic differential defining that ray. Globally, we show that the corresponding family of equivariant harmonic maps to a symmetric space converge to a harmonic map into the asymptotic cone of that space. The geometry of the image may also be described by that differential: it is weakly convex and a (one-third) translation surface. We define a compactification of the Hitchin component in this setting for triangle groups that respects the parametrization by Hitchin differentials.

Contents

1. Introduction

In a pioneering work in the 1980's and early 1990's, Hitchin and others ([Hit87], [Cor88], [Sim92]) developed a beautiful non-Abelian Hodge theory for character varieties of surface groups in higher rank Lie groups. The subject has been intensely studied ever since and while new perspectives, for example more synthetic/algebro-geometric [FG06] or dynamical [Lab06, Gui08], have emerged, there remain some basic questions about how to properly geometrically interpret Hitchin's original parametrizaton of a principal component of the character variety in terms of the holomorphic data he used. (Indeed, in [Hit92], Hitchin remarks, "Unfortunately, the analytical point of view used for the proofs gives no indication of the *geometrical*

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significance of the Teichmüller component.") The goal of this paper is to relate the holonomy of a representation in a Hitchin component to the local synthetic geometry of the holomorphic differential that Hitchin associates to it, at least in an asymptotic sense.

More precisely, we focus on the Hitchin component Hit_3 of surface group representations into $\mathrm{SL}(3,\mathbb{R})$. Hitchin ([Hit92]) parametrizes this component in terms of pairs (q_2, q_3) of quadratic and cubic differentials $q_2 \in H^0(X_0, K_{X_0}^2), q_3 \in H^0(X_0, K_{X_0}^3)$ on a fixed Riemann surface X_0. A parametrization, invariant under the action of the mapping class group, was given independently by Labourie ([Lab07a]) and Loftin ([Lof01]): from their perspective, Hit_3 may be seen as the cubic differential bundle $\mathcal{C}$ over the Teichmüller space $\mathcal{T}(S)$.

We study families defined by rays in this parametrization, and in particular the asymptotics. Of course, a ray is defined as multiples sq_0, for q_0 a fixed cubic differential on a fixed Riemann surface $\Sigma = (S, J)$ and $s > 0$, and so has holomorphic invariants that are projectively fixed; on the other hand, via the Labourie-Loftin parametrization, the ray defines a family of holonomies Hol_s of representations ρ_s. We relate the asymptotics of the holonomies to the holomorphic invariants of the form q_0: we give a formula for the leading term of the holonomy $\mathrm{Hol}_s([\gamma])$ of a curve class $[\gamma]$ in terms of the intersection number of $[\gamma]$ with the form q_0. The class $[\gamma]$ may be represented by a geodesic c_γ in the metric $|q_0|^{\frac{2}{3}}$: this metric is flat away from the zeroes of q_0, with cone points of total angle $2\pi(1+\frac{1}{3})\deg_p(q_0)$ at a zero p of q_0. The segments, known as *saddle connections*, between the zeroes are denoted $c_1, ..., c_l$.[1] Of course, in such a zero-free region, the cubic differential q_0 has three well-defined cube roots ϕ_1, ϕ_2, ϕ_3 and we may compute intersection numbers $-2^{\frac{2}{3}}\mathcal{R}e\int_\delta \phi_j$ of curves δ against these roots. Let ν^i denote the largest of the real parts of the three periods along c_i; this is equivalent to the logarithm of the largest eigenvalue of the holonomy of a natural development of that saddle connection into affine space. Then our main results on asymptotic holonomy may be summarized in the theorem below: we comment later on what is elided in the statement as well as some of the subtleties in its statement and hence proof.

Theorem A. *For every curve class* $[\gamma]$*, we have*

$$\lim_{s\to+\infty} \frac{\log \|\mathrm{Hol}_s([\gamma])\|}{s^{\frac{1}{3}}} = \sum_{i=1} \nu^i, \tag{1.1}$$

where $\|\cdot\|$ *is any submultiplicative matrix norm.*

In particular, if $\sigma_j(\mathrm{Hol}_s(c_\gamma))$ *denotes the* j*-th largest singular value of* $\mathrm{Hol}_s(c_\gamma)$*, where* c_γ *is a flat geodesic in the homotopy class of* γ *with saddle connections* $c_1, \dots, c_l$*, then*

$$\lim_{s\to+\infty} \frac{\log(\sigma_j(\mathrm{Hol}_s(c_\gamma)))}{s^{\frac{1}{3}}} = \sum_{i=1}^{l} \lim_{s\to+\infty} \frac{\log(\sigma_j(\mathrm{Hol}_s(c_i)))}{s^{\frac{1}{3}}} \tag{1.2}$$

[1]Saddle connections in this paper are just Euclidean geodesic segments, not restricted to be horizontal in some way.

for $j = 1, 2, 3$.

As suggested previously, the main qualitative result is that the leading term of the holonomy, in this regime of a ray, is visible through the local expressions of the Hitchin holomorphic parametrization. That the right-hand-side of (1.1) is expressed as a sum of maxima is consistent with other "tropical" expressions regarding asymptotics, see for example [Foc98].

There are some nuances. First, we observe the role of the zeros of the holomorphic form q_0. They have no explicit presence in either formula (1.1) or (1.2).

Yet, there is a subtlety in that we add the dominant eigenvalue for each saddle connection, so these must get aligned as the geodesic c_γ transitions from a saddle connection coming into a zero. Here, two considerations collide: first we must understand the general form of the unipotents that arise as a path crosses the Stokes lines that emanate from a zero [DW15]. Second, we must show that the matrix permutations defined by these unipotents match up the dominant eigenvalues of saddle connections: this occurs because of the geometry of the unipotents that occur in the limits. The resulting agreement of directions between incoming and outgoing saddle connections is a linchpin of the current work. It is somewhat remarkable that it holds not only generically but also in the special cases where the saddle connections are in the directions of Stokes lines or walls of Weyl chambers.

We also note that, in contrast to some treatments (e.g. [MSWW16], [OSWW20]), we do not restrict to simple zeroes. In general, this present formula might be seen as an extension of the work of the first author in [Lof07]. (See also the extension of that work by Collier-Li [CL17] on more general cyclic Higgs bundles.)

Theorem A also has an interpretation in terms of the Weyl-chamber length compactification of the Hitchin component introduced by Parreau ([Par12]). Let $\mathfrak{a}^+$ be the positive Weyl-chamber in $\mathfrak{sl}(3,\mathbb{R})$ given by diagonal matrices where entries appear in decreasing order. To a Hitchin representation $\rho : \pi_1(\Sigma) \to \mathrm{SL}(3,\mathbb{R})$, Parreau associated an $\mathfrak{a}^+$-valued length spectrum $\{L_\rho(\gamma)\}_{\gamma\in\pi_1(\Sigma)}$ by considering the Jordan projection of $\rho(\gamma)$, which represents geometrically the (vector-valued) translation length of each element $\rho(\gamma)$ acting on the symmetric space $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$. The Weyl-chamber length compactification $\overline{\mathrm{Hit}_3}^{WL}$ is then defined as the closure of the image of the map

$$\begin{aligned} j : \mathrm{Hit}_3 &\to \widehat{\mathrm{Hit}_3} \times \mathbb{P}(\overline{\mathfrak{a}^+}^{\pi_1(S)}) \\ \rho &\mapsto (\rho, L_\rho(\cdot)) \ , \end{aligned}$$

where $\widehat{\mathrm{Hit}_3}$ is the one point compactification of Hit_3. Our Theorem A thus implies the following:

Corollary B. *Let ρ_s be a diverging family of Hitchin representations corresponding to a ray of cubic differentials $q_s = sq_0$ in the Labourie-Loftin parameterization of* Hit_3. *Then ρ_s converges to a unique point L_∞ in $\partial\overline{\mathrm{Hit}_3}^{WL}$, which can be computed explicitly as follows: if the geodesic representative in the homotopy class of γ is given* *by a concatenation of saddle connections $c_1, \ldots, c_l$, then*

$$L_\infty(\gamma) = \sum_{i=1}^{l} \left(-2^{\frac{2}{3}} \mathcal{R}e \int_{c_i} \phi_1, -2^{\frac{2}{3}} \mathcal{R}e \int_{c_i} \phi_2, -2^{\frac{2}{3}} \mathcal{R}e \int_{c_i} \phi_3 \right)$$

where ϕ_1, ϕ_2 and ϕ_3 are the cube roots of q_0 and are numbered on each c_i so that the entries of the vector above appear in non-increasing order.

It is important to note that, since $\|L_\infty(\gamma)\|_\infty$ is, up to a multiplicative constant, the length of the geodesic representative of γ for the flat metric $|q_0|^{\frac{2}{3}}$, the infimum $\inf_\gamma \|L_\infty(\gamma)\|_\infty$ is strictly positive, hence L_∞ lies in the domain of discontinuity for the action of the mapping class group on $\overline{\mathrm{Hit}_3}^{WL}$ ([BIPP21a]).

A similar result about the asymptotic behavior of length functions for Hitchin representations associated to rays of holomorphic cubic differentials has also been recently found by Reid [Rei23].

Finally, we relate these considerations to the harmonic map $h_s \colon \tilde{\Sigma} \to X$ defined via the solution to Hitchin's equations. Now, equation (1.2) gives asymptotics of the singular values, and we recall that the singular values of an element $M \in \mathrm{SL}(3, \mathbb{R})$ define the distance in the symmetric space $X = \mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3)$ between the origin and $[M]$, with asymptotics then defining distance in the asymptotic cone. We then study the geometry of the limiting harmonic map to the asymptotic cone, obtained by taking an ω-limit of X, rescaling by the growth of the co-diameter of the image of h_s. In his thesis ([Kim17]), Semin Kim proves a more general version of the following.

Proposition 1.1 ([Kim17])**.** *Upon rescaling by $s^{-\frac{1}{3}}$, the family of harmonic maps $h_s : \tilde{\Sigma} \to X$ converges to a Lipschitz harmonic map $h_\infty : \tilde{\Sigma} \to \mathrm{Cone}_\omega(X)$. The corresponding rescaled family of holonomy representations $\rho_s : \pi_1(\Sigma) \to \mathrm{SL}(3, \mathbb{R})$ converges to a representation ρ_∞ to the isometry group of $\mathrm{Cone}_\omega(X)$. The harmonic map h_∞ is equivariant with respect to ρ_∞.*

We give a more precise description of the image of the limiting harmonic map h_∞:

Theorem C. *The cubic differential q_0 induces a $\frac{1}{3}$-translation surface structure on the image $h_\infty(\tilde{\Sigma}) \subset \mathrm{Cone}_\omega(X)$, which is compatible with the local geometry of $\mathrm{Cone}_\omega(X)$.*

More precise statements and proofs are contained in Proposition 8.6 and Theorems 8.7 and 8.10 below.

The structure on $h_\infty(\tilde{\Sigma})$ involves natural local models $u_k : B \to (X, d)$, where B is a ball and u_k is a conformal harmonic map to the asymptotic cone $\mathrm{Cone}_\omega(X)$ defined in terms of the cubic differential $\psi_k = z^k dz^3$ and its three cube roots ϕ_1, ϕ_2, ϕ_3. In particular, the image of the punctured ball $B \setminus \{0\}$ comprises $2(k+3)$ flat sectors W_i which meet only consecutively on geodesics and to which u_k is defined by

$$u_k(x) = \left(-2^{\frac{2}{3}} \mathcal{R}e \left(\int_0^x \phi_1 \right), -2^{\frac{2}{3}} \mathcal{R}e \left(\int_0^x \phi_2 \right), -2^{\frac{2}{3}} \mathcal{R}e \left(\int_0^x \phi_3 \right) \right)$$

for x in a sector of B. We prove in Theorems 8.7 and 8.10 that the harmonic map h_∞ has this local structure, where ψ_k is a restriction of q_0 to B.

Although the asymptotic cone $\mathrm{Cone}_\omega(X)$, the limiting harmonic map h_∞ and the representation ρ_∞ depend heavily on the choice of the ultrafilter ω, the geometry of the image $h_\infty(\tilde{\Sigma})$ and, in particular, the fact that it carries a 1/3-translation surface structure induced by the cubic differential q_0 is independent of all choices. This may be compared to the classical setting of Teichmüller space, where equivariant harmonic maps $u_s : \tilde{\Sigma} \to \mathbb{H}^2$ are parameterized by a family of quadratic differentials Q_s. In that case, if $Q_s = sQ_0$ is a ray, u_s always converges to an equivariant harmonic map $u_\infty : \tilde{\Sigma} \to T$ to a real tree given by projection onto the leaves of the vertical foliation of Q_0 ([Wol95]). The projective class of the vertical foliation only depends on Q_0 and not on the particular diverging sequence. Our result suggests that a harmonic map compactification of Hit_3 analogous to the harmonic maps compactification of Teichmüller space should include in its boundary projective classes of 1/3-translation surfaces.

Now, group actions on buildings have arisen in the work of several authors ([Par12], [BIPP21b]) with harmonic maps to these buildings having some prominence ([BIPP21b], [KNPS15], [MOT21]). Here one might compare the lower rank constructions of surface group actions on real trees in the context of $\mathrm{SL}(2,\mathbb{R})$ character varieties: see [Bes88], [Wol89], [Wol95]. In the present context, it is worth focusing on the papers of Katzarkov-Noll-Pandit-Simpson ([KNPS15], [KNPS17]), in which the authors outline an approach to compactifying character varieties of surface group representations in $\mathrm{SL}(d,\mathbb{C})$. Of course, the present paper can be seen as demonstrating a part of that program in a real setting.

Moreover, though, in a companion paper ([LTW25]), we prove a uniqueness theorem for conformal equivariant harmonic maps to buildings which applies in our situation. Thus we find that in settings in which the harmonic maps have an image in the asymptotic cone of $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$, those (equivariant) harmonic maps coincide with the maps described in this paper as endpoints of rays. In particular, those maps would be definable in terms of cubic differentials projectively approximated by the Hitchin differentials for the approximating representations. The results of Theorems 8.7 and 8.10, and the uniqueness results in [LTW25], then in some sense unify some of the various approaches to asymptotic holonomy of representations and the limiting buildings.

In the concluding section of this paper, we provide an example of a full compactification in a specific example, that of the $\mathrm{SL}(3,\mathbb{R})$ Hitchin component of most (p,q,r)-triangle groups.

Two features of our technique limit the scope of these results but one extends it in an important way beyond existing results. First, we rely heavily on the fact that the Higgs bundles associated to these representations are cyclic, and the resulting substantial symmetries in the Hitchin system. Indeed, in the present work, that system is but a single scalar equation. That is somewhat less of a limitation than it may seem, as some analogous results are available in the case of $\mathrm{Sp}(4,\mathbb{R})$ and G_2' ([OT23], [TW24], [Eva22]). Second, here we fix the conformal structure of the domain: considering asymptotics where both the domain Riemann surface and the representation degenerate seems to require an analysis finer than what we present here. Finally, we are greatly aided by the asymptotic cone being two-dimensional, and hence the same dimension as the domain Σ. This restricts the flexibility of the limiting harmonic maps.

On the other hand, in contrast to all previous works on the asymptotics of Hitchin representations along rays of cubic differentials (e.g. [CL17], [Lof07], [Moc16]), our techniques allow to handle *all* paths on Σ, including those represented by singular geodesics for the flat metric $|q_0|^{\frac{2}{3}}$ through zeros of any order, and give a local geometric description of the limiting harmonic map to the building also around the zeros of the cubic differential.

Organization. In the second section, we define our notation and present some background material. Section 3 is devoted to presenting some required analysis of the Hitchin partial differential equation which governs the harmonic maps. In Section 4, we analyze the holonomy near a zero of q_0: we are interested in the holonomy along a generic saddle connection and along a portion of an arc that links the zero that crosses a Stokes line. Section 5 assembles these partial holonomies of individual saddle connections into a preliminary description of the asymptotic holonomy. Then, in Section 6, we discuss the phenomenon that the unipotents that describe the transition between incoming and outgoing saddle connections intertwine the dominant eigenvalues. In Section 7, we collect all of the ingredients from the previous sections and prove the main result on asymptotic holonomy displayed above. Section 8 pivots to describe how the endpoint of a ray is a harmonic map to a building, displaying the local structure and induced metric to q_0, and showing that the image is "weakly convex" in the sense of Parreau [Par22]. Finally, Section 9 displays a corollary of our work in a very special case: the compactification of Hit_3 in the case of a triangle group, where we can provide a somewhat complete account of the compactification using our methods.

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2. Background material

The geometry relating cubic differentials and representations in Hit_3 is that of a special sort of surface, a hyperbolic affine sphere H, in $\mathbb{R}^3$. We briefly sketch the elements of the theory we will need: for a more detailed account, the reader might consult [Lof10], [LM16], [Lab07b], [Wan91], [Bar15], [BH13].

There are natural affine invariants on a hyperbolic affine sphere, a Riemannian metric called the Blaschke metric g, and a cubic differential q which is holomorphic with respect to the conformal structure induced by the Blaschke metric, related by the compatibility relation

$$\kappa(g) = -1 + 2\|q\|_g^2, \tag{2.1}$$

where $\kappa(g)$ is the Gauss curvature, and $\|q\|_g$ is the pointwise norm of q with respect to the metric g. Let $f\colon \mathcal{D} \to \mathbb{R}^3$ be a parametrization of H conformal with respect to g, from a domain $\mathcal{D} \subset \mathbb{C}$. Then f is transverse to the tangent plane $T_f H$ and we have the following structure equations for the frame $F = (f\ f_z\ f_{\bar{z}})$ in $\mathbb{R}^3$, for z a conformal coordinate and $g = e^{\phi}|dz|^2$:

$$F^{-1}dF = \begin{pmatrix} 0 & 0 & \frac{1}{2}e^{\phi} \\ 1 & \partial_z\phi & 0 \\ 0 & qe^{-\phi} & 0 \end{pmatrix} dz + \begin{pmatrix} 0 & \frac{1}{2}e^{\phi} & 0 \\ 0 & 0 & \bar{q}e^{-\phi} \\ 1 & 0 & \partial_{\bar{z}}\phi \end{pmatrix} d\bar{z}. \tag{2.2}$$

Equation (2.2) then gives the connection form of the pullback ∇ of the flat connection on $\mathbb{R}^3$ to the rank-3 bundle $\mathbb{1} \oplus T_{\mathbb{C}}\mathcal{D}$ over $\mathcal{D}$, for $\mathbb{1}$ the trivial line bundle. Equation (2.1) and $\partial_{\bar{z}} q = 0$ ensure ∇ is flat. We will be interested in the case where $\mathcal{D}$ is a disk representing the universal cover of a closed Riemann surface, and the fundamental group of the surface acts equivariantly on the flat bundle $(\mathbb{1} \oplus T_{\mathbb{C}}\mathcal{D}, \nabla)$. That bundle $\mathbb{1} \oplus T_{\mathbb{C}}\mathcal{D}$ then descends to a flat vector bundle $\mathbb{1} \oplus T_{\mathbb{C}}\Sigma$ over Σ, whose holonomy gives a Hitchin representation into $\mathrm{SL}(3,\mathbb{R})$. We refer to this setting in our considerations of Section 5; see in particular Remark 5.1.

Alternately, given a holomorphic cubic differential q over Σ, we may use this data to define a stable rank-3 Higgs bundle, which then induces an equivariant conformal harmonic map from $\tilde{\Sigma}$ to the symmetric space $X = \mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$. We need not avail ourselves of the details of this construction, as the harmonic map can be constructed directly from the hyperbolic affine sphere. To see this, identify X with the space of all metrics (positive-definite quadratic forms) on $\mathbb{R}^3$ of determinant 1. The Blaschke lift h ([Lab07a], [Lof04], [BP94]) at a given point f on H is then the metric on $\mathbb{R}^3$ for which

- f has norm 1,
- f is orthogonal to $T_f H$,
- the metric restricted to $T_f H$ is the Blaschke metric g.

To relate the Blaschke lift h to the frame F, it is useful to use an orthonormal frame for h. For $z = x + iy$, the frame

$$\hat{F} = \begin{pmatrix} f & \frac{f_x}{|f_x|_g} & \frac{f_y}{|f_y|_g} \end{pmatrix} = F \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-\phi/2} & ie^{-\phi/2} \\ 0 & e^{-\phi/2} & -ie^{-\phi/2} \end{pmatrix}$$

is orthonormal. Thus

$$h = (\hat{F}^\top)^{-1} \hat{F}^{-1} = (F^\top)^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2}e^{\phi} \\ 0 & \frac{1}{2}e^{\phi} & 0 \end{pmatrix} F^{-1}.$$

Cheng-Yau ([CY86], [CY77]) proved that any hyperbolic affine sphere with a complete Blaschke metric is asymptotic to a properly convex cone in $\mathbb{R}^3$, which we take to have vertex at the origin. Likewise, for every such convex cone, there is a unique hyperbolic affine sphere asymptotic to the cone invariant under special linear automorphisms of the cone and with complete Blaschke metric. By projecting $\mathbb{R}^3 \to \mathbb{RP}^2$, we may identify H with a properly convex domain in $\mathbb{RP}^2$. On a compact Riemann surface Σ of genus at least 2 equipped with a cubic differential q, there is a unique solution to Equation (3.1) below (this is the global version of Equation (2.1) above). Thus the universal cover $\tilde{\Sigma}$ is identified with a convex domain Ω in $\mathbb{RP}^2$, and Σ itself admits a quotient of Ω by projective automorphisms. In other words, we have induced a convex $\mathbb{RP}^2$ structure on Σ. Choi-Goldman ([Gol90],[CG93]) show that convex $\mathbb{RP}^2$ structures are equivalent to Hitchin representations.

Dumas and the third author [DW15] address the case of polynomial cubic differentials on $\mathbb{C}$, and show there is a unique complete Blaschke metric on that plane. The convex domain in this case is a convex polygon with $d + 3$ sides, for d the degree of the polynomial. Indeed polynomial cubic differentials on $\mathbb{C}$ are equivalent to convex polygons, up to appropriate equivalences. In this work, we are primarily interested in cubic differentials of the form $z^d \, dz^3$, which corresponds to the regular convex polygon of $d + 3$ sides. In general, the analysis of the boundary of the polygon follows by comparison to the simpler geometry of inscribed and circumscribed triangles around each vertex and edge.

Example 2.1. The triangle case (for a constant cubic differential on $\mathbb{C}$ and in which the Blaschke metric is flat) can be worked out explicitly and goes back to Ţiţeica ([Tzi08]). We see that this is a fundamental model for us, and so we describe some of its features. In terms of a natural local coordinate in which the cubic differential $q = dw^3$, for $w = re^{i\theta}$, there are "Stokes" directions for rays $\theta = \frac{\pi}{6} + \frac{\pi}{3}k$ for k an integer. In the sectors bounded by $\theta = \frac{\pi}{3} + \frac{2\pi}{3}k$, the rays limit on a vertex of the triangle; the rays parallel to those angles fill out the sides of the polygon.

For a general convex polygon, in each sector in between the Stokes directions, the affine sphere is well approximated by an appropriate Ţiţeica surface. Upon crossing a Stokes ray, the approximating Ţiţeica surface changes by the action of a unipotent transformation determined by the geometry of the convex polygon: the unique unipotent transformation in $\mathbb{R}^3$ whose projective action on $\mathbb{RP}^2$ transforms a given inscribed (circumscribed) triangle in the polygon to the subsequent circumscribed (inscribed) triangle by moving a single vertex.

We will also encounter other special directions in $\mathbb{C}$, which represent the walls of Weyl chambers at $\theta = \frac{\pi}{3}k$, upon identifying $\mathbb{C}$ with the maximum torus of the Lie algebra of $\mathrm{SL}(3,\mathbb{R})$.

3. Asymptotics of the Blaschke metric

Consider a ray of cubic differentials $q_s = sq_0$ on a fixed closed Riemann surface $\Sigma = (S, J)$ with conformal hyperbolic metric σ. Let $g_s = e^{\mu_s}\sigma$ be the Blaschke metric on the associated affine sphere. The real-valued functions μ_s are solutions of Wang's equation

$$\Delta_\sigma \mu_s = 2e^{\mu_s} - 4e^{-2\mu_s}\frac{|q_s|^2}{\sigma^3} + 2\kappa(\sigma). \tag{3.1}$$

Near each zero of q, Nie has studied rescaled limits of the associated convex $\mathbb{RP}^2$ structure, which converge to a regular polygon [Nie22]. Our approach focuses on precise estimates comparing the corresponding affine spheres.

3.1. Asymptotics far from the zeros.

We compare the Blaschke metrics g_s with the flat metric with cone singularities $|q_s|^{\frac{2}{3}}$. The estimates in this subsection are well-known and indeed hold in far greater generality (cf. [Moc16]), but we restrict here to our present setting, where some of the estimates can be more explicit.

Lemma 3.1 ([Lof04], [DW15])**.** *The Blaschke metric g_s satisfies $g_s > 2^{\frac{1}{3}}|q_s|^{\frac{2}{3}}$.*

Lemma 3.2 ([OT21],[Moc16] Prop. 2.1)**.** *The area of the Blaschke metric satisfies*

$$2^{\frac{1}{3}}\|q_s\| \leq \mathrm{Area}(S, g_s) \leq 2^{\frac{1}{3}}\|q_s\| + 2\pi|\chi(S)|$$

where $\|q\| = \int_S |q|^{\frac{2}{3}}$.

We consider now the quantity

$$\mathcal{F}_s = \mu_s - \frac{1}{3}\log\left(\frac{2|q_s|^2}{\sigma^3}\right)$$

in order to compare g_s and $|q_s|^{\frac{2}{3}}$ outside the zeros of q_s. Outside the zeros of q_0, the function $\mathcal{F}_s$ satisfies the PDE

$$\Delta_{|q_0|^{\frac{2}{3}}}\mathcal{F}_s = 2^{\frac{4}{3}}s^{\frac{2}{3}}(e^{\mathcal{F}_s} - e^{-2\mathcal{F}_s}).$$

By Lemma 3.1, $\mathcal{F}_s > 0$ and $\Delta_\sigma \mathcal{F}_s > 0$. Hence $\mathcal{F}_s$ is subharmonic.

Lemma 3.3 (Coarse bound on $\mathcal{F}_s$). *Let $p \in S$ and let r_0 be the radius of a ball around p for the flat metric $|q_0|^{\frac{2}{3}}$ which does not contain any zeros of q_0. Then*

$$\mathcal{F}_s(p) \le \log\left(\frac{\text{Area}(S, g_s)}{2^{\frac{1}{3}}\pi s^{\frac{2}{3}} r_0^2}\right).$$

Proof. The ball B of radius $s^{\frac{1}{3}} r_0$ for the flat metric q_s does not contain any zeros. By subharmonicity of $\mathcal{F}_s$ and Jensen's Inequality, we have

$$e^{\mathcal{F}_s} \le e^{\fint_B \mathcal{F}_s dA_{q_s}} \le \fint_B e^{\mathcal{F}_s} dA_{q_s} = 2^{-\frac{1}{3}} \fint_B e^{\mu_s} dA_\sigma \le \frac{\text{Area}(S, g_s)}{2^{\frac{1}{3}}\pi s^{\frac{2}{3}} r_0^2}.$$

□

Lemma 3.4 (Error decay). *Let $p \in S$ be a point at distance r_0 from the zeros of q_0 for the singular flat metric $|q_0|^{\frac{2}{3}}$. Then for any $\delta > 1$, there is a $D > 0$ so that*

$$\mathcal{F}_s(p) \le D s^{\frac{1}{6}} e^{-\sqrt{3}\cdot 2^{\frac{2}{3}} a_s s^{\frac{1}{3}} r_0/\delta},$$

with $a_s \to 1$ as $s \to +\infty$.

Proof. Consider the ball centered at p of radius $s^{\frac{1}{3}} r_0/\delta$ for the flat metric q_s. Since this ball does not contain any zeros of q_0, we may choose coordinates on the ball so that $q_0 = dz^3$ and p is at $z = 0$. Then $\mathcal{F}_s$ on this ball satisfies

$$\Delta \mathcal{F}_s = 2^{\frac{4}{3}} s^{\frac{2}{3}} \cdot 2e^{-\mathcal{F}_s/2} \sinh\left(\frac{3}{2}\mathcal{F}_s\right).$$

By Lemma 3.3 and Lemma 3.2, the function $e^{-\mathcal{F}_s/2}$ is uniformly bounded below by a constant $c > 0$. Then

$$\Delta \mathcal{F}_s \ge 3 \cdot 2^{\frac{4}{3}} s^{\frac{2}{3}} c \mathcal{F}_s.$$

Similarly, the function $\mathcal{F}_s$ is bounded above by a constant $A = A(\delta) > 0$ on the boundary of the ball. Let η be the solution of the system

$$\begin{cases} \Delta \eta = 3 \cdot 2^{\frac{4}{3}} s^{\frac{2}{3}} c \eta, \\ \eta|_\partial = A. \end{cases}$$

We know $\eta(r) = A \frac{I_0(\sqrt{3}\cdot 2^{\frac{2}{3}}\sqrt{c}\cdot s^{\frac{1}{3}} r)}{I_0(\sqrt{3}\cdot 2^{\frac{2}{3}}\sqrt{c}\cdot s^{\frac{1}{3}} r_0/\delta)}$, where $I_0(\kappa r)$ is the only radial solution of

$$\begin{cases} \Delta I_0 = \kappa^2 I_0, \\ I_0|_\partial = 1. \end{cases}$$

It is well-known ([AS64]) that $I_0(\kappa x)$ is the Bessel function of the first kind and has asymptotic behavior $I_0(\kappa x) \sim \frac{e^{\kappa x}}{\sqrt{x}}$ as $x \to +\infty$. By the maximum principle,

$$\mathcal{F}_s(0) \le \eta(0) = \frac{A}{I_0(\sqrt{3}\cdot 2^{\frac{1}{3}}\sqrt{c}\cdot s^{\frac{1}{3}} r_0/\delta)} \le D s^{\frac{1}{6}} e^{-\sqrt{3}\cdot 2^{\frac{2}{3}}\sqrt{c}\cdot s^{\frac{1}{3}} r_0/\delta}.$$

This implies that $\mathcal{F}_s$ decays exponentially outside the zeros of q_0. Now, remember that the constant c comes from the bound on $e^{-\mathcal{F}_s/2}$. From the decay of $\mathcal{F}_s$, we can improve this bound to $e^{-\mathcal{F}_s/2} \geq a_s$ with $a_s \to 1$ as $s \to +\infty$, and the lemma is proved. □

Notation. For simplicity, we denote by $m(s)$ the exponent appearing in Lemma 3.4, i.e.

$$m(s) = m(s, r_0, \delta) = \sqrt{3} \cdot 2^{\frac{2}{3}} a_s s^{\frac{1}{3}} r_0/\delta.$$

3.2. **Estimates around a zero.** We now move to the study of the asymptotic behavior of μ_s around a zero of the Pick differential. We denote by $h_s = e^{\nu_s}|dz|^2$ the Blaschke metric of the affine sphere with polynomial cubic differential $q_s = sz^k dz^3$ on $\mathbb{C}$. We want to compare μ_s and ν_s on the ball $B = \{|z| < \epsilon\}$. We rewrite the Blaschke metric $g_s = e^{\mu_s}\sigma$ with respect to the background flat metric $|dz|^2$ on B: $g_s = e^{\phi_s}|dz|^2$, where ϕ_s is a solution of the PDE (only defined on the closure of B)

$$\Delta\phi_s = 2e^{\phi_s} - 4e^{-2\phi_s}|q_s|^2. \tag{3.2}$$

Thus ν_s is a solution of the same equation as ϕ_s, but with different boundary values.

Lemma 3.5. *On ∂B, we have $\phi_s - \nu_s = O(s^{\frac{1}{6}} e^{-m(s)})$ as $s \to +\infty$.*

Proof. We know from Lemma 3.4 that

$$\phi_s|_{\partial B} = \mu_s + \log(\sigma) = \frac{1}{3}\log(2s^2\epsilon^{2k}) + O(s^{\frac{1}{6}} e^{-m(s)})$$

for $r_0 = \frac{3}{k+3}\epsilon^{\frac{k+3}{3}}$, where we recall that σ is the hyperbolic metric on the fixed Riemann surface Σ. Thus it is sufficient to show that

$$\nu_s = \frac{1}{3}\log(2s^2\epsilon^{2k}) + o(s^{\frac{1}{6}} e^{-m(s)}).$$

Changing complex coordinates to $w = s^{\frac{1}{k+3}} z$, the ball B can be rewritten as $B = \{|w| < s^{\frac{1}{k+3}}\epsilon\}$. In these coordinates, $e^{\nu_s}|dz|^2 = e^{\psi_s}|dw|^2$ with $\psi_s = \nu_s - \frac{2}{k+3}\log(s)$, and $q_s = w^k dw^3$. The estimates of ([DW15, Theorem 5.7]) then tell us that

$$\psi_s|_{\partial B} = \frac{1}{3}\log(2s^{\frac{2k}{k+3}}\epsilon^{2k}) + O\left(\frac{e^{-m(s)}}{s^{\frac{1}{6}}}\right).$$

Hence, on ∂B,

$$\begin{aligned}
\nu_s &= \psi_s + \frac{2}{k+3}\log(s) \\
&= \frac{1}{3}\log(2s^{\frac{2k}{k+3}}\epsilon^{2k}) + \frac{1}{3}\log(s^{\frac{6}{k+3}}) + O\left(\frac{e^{-m(s)}}{s^{\frac{1}{6}}}\right) \\
&= \frac{1}{3}\log(2s^2\epsilon^{2k}) + O\left(\frac{e^{-m(s)}}{s^{\frac{1}{6}}}\right).
\end{aligned}$$

□

Lemma 3.6. *On the ball B, we have $\phi_s - \nu_s = O(s^{\frac{1}{6}} e^{-m(s)})$ as $s \to +\infty$.*

Proof. Define $\eta_s = \phi_s - \nu_s$. It satisfies the PDE

$$\Delta\eta_s = \Delta\phi_s - \Delta\nu_s = 2e^{\phi_s} - 4e^{-2\phi_s}|q_s|^2 - 2e^{\nu_s} + 4e^{-2\nu_s}|q_s|^2$$

on B. Dividing by e^{ν_s}, we find, upon setting $h_s = e^{\nu_s}|dz|^2$, that

$$\begin{aligned}\Delta_{h_s}\eta_s &= e^{-\nu_s}\Delta\eta_s = 2e^{\eta_s} - 4e^{-2\phi_s-\nu_s}|q_s|^2 - 2 + 4e^{-3\nu_s}|q_s|^2 \\ &= 2(e^{\eta_s}-1) - 4\frac{|q_s|^2}{e^{3\nu_s}}(e^{-2\eta_s}-1).\end{aligned}$$

By the maximum principle,

$$|\eta_s| \le \max(|\max(\eta_s|_{\partial B})|, |\min(\eta_s|_{\partial B})|),$$

which gives the desired estimate by Lemma 3.5. □

4. Comparison between affine spheres

We denote by $F_s(z)$ the frame field of the affine sphere arising from the data (σ, sq_0) on the surface S restricted to the disk $B = \{|z| < \epsilon\}$. We normalize the affine sphere so that $F_s(0) = \mathrm{Id}$ for all $s > 0$. With this choice F_s will be complex-valued, belonging to a subgroup of $\mathrm{SL}(3,\mathbb{C})$ isomorphic to $\mathrm{SL}(3,\mathbb{R})$. Recall that F_s is the solution to the ODE

$$\begin{cases} F_s^{-1}dF_s = U_s dz + V_s d\bar{z}, \\ F_s(0) = \mathrm{Id}, \end{cases}$$

where U_s, V_s are the matrices arising from the structure equations of the affine sphere. Precisely,

$$U_s = \begin{pmatrix} 0 & 0 & \frac{1}{2}e^{\phi_s} \\ 1 & \partial_z\phi_s & 0 \\ 0 & q_s e^{-\phi_s} & 0 \end{pmatrix}, \qquad V_s = \begin{pmatrix} 0 & \frac{1}{2}e^{\phi_s} & 0 \\ 0 & 0 & \bar{q}_s e^{-\phi_s} \\ 1 & 0 & \partial_{\bar{z}}\phi_s \end{pmatrix}. \tag{4.1}$$

We denote by $F_M(w)$ the frame field of the (model) affine sphere over $\mathbb{C}$ with polynomial cubic differential $w^k dw^3$ normalized so that $F_M(0) = \mathrm{Id}$. Note $F_M(w)$ solves

$$\begin{cases} F_M^{-1}dF_M = U_M dw + V_M d\bar{w}, \\ F_M(0) = \mathrm{Id}. \end{cases}$$

We compare F_s and F_M on the disk $B = \{|z| < \epsilon\}$ centered at a zero of order k for q_0. Recall that $w = s^{\frac{1}{k+3}}z$, so we consider

$$G_s(z) = F_s(z)F_M^{-1}(s^{\frac{1}{k+3}}z).$$

The matrices G_s satisfy the differential equation

$$\begin{aligned} G_s^{-1}dG_s &= F_M(s^{\frac{1}{k+3}}z)F_s^{-1}(z)[dF_s\cdot F_M^{-1}(s^{\frac{1}{k+3}}z) - F_s(z)F_M^{-1}(s^{\frac{1}{k+3}}z)dF_M F_M^{-1}(s^{\frac{1}{k+3}}z)s^{\frac{1}{k+3}}] \\ &= F_M(s^{\frac{1}{k+3}}z)[U_s dz + V_s d\bar{z} - U_M dz - V_M d\bar{z}]F_M^{-1}(s^{\frac{1}{k+3}}z) \\ &= F_M(s^{\frac{1}{k+3}}z)[(U_s - U_M)dz + (V_s - V_M)d\bar{z}]F_M^{-1}(s^{\frac{1}{k+3}}z) \\ &= F_M(s^{\frac{1}{k+3}}z)\Theta_s F_M^{-1}(s^{\frac{1}{k+3}}z), \end{aligned}$$

when written in the z coordinate, where

$$\Theta_s(z) = \begin{pmatrix} 0 & 0 & (e^{\phi_s} - e^{\nu_s})/2 \\ 0 & \partial_z(\phi_s - \nu_s) & 0 \\ 0 & sz^k(e^{-\phi_s} - e^{-\nu_s}) & 0 \end{pmatrix} dz + \begin{pmatrix} 0 & (e^{\phi_s} - e^{\nu_s})/2 & 0 \\ 0 & 0 & s\bar{z}^k(e^{-\phi_s} - e^{-\nu_s}) \\ 0 & 0 & \partial_{\bar{z}}(\phi_s - \nu_s) \end{pmatrix} d\bar{z}.$$

and ν_s was defined in Section 3.2.

Lemma 4.1. *On B, we have $\|\Theta_s\|_\infty \le Cs^{\frac{5}{3}}e^{-m(s)}$ as $s \to +\infty$.*

Proof. We handle the various entries in Θ_s one-by-one. First of all, the Mean Value Theorem implies for $|z| < \epsilon$ that

$$e^{\phi_s(z)} - e^{\nu_s(z)} = e^{p(z)}(\phi_s(z) - \nu_s(z))$$

for some $p(z)$ between $\phi_s(z)$ and $\nu_s(z)$. A straightforward application of the Maximum Principle applied to Equation (3.2) on B, together with a boundary estimate from Lemma 3.4, then gives

$$\phi_s(z) \le \frac{1}{3}\log(2s^2\epsilon^{2k}) + o(1),$$

and the same is true for $\nu_s(z)$. Then Lemma 3.6 shows

$$e^{\phi_s(z)} - e^{\nu_s(z)} = O(s^{\frac{2}{3}} \cdot s^{\frac{1}{6}}e^{-m(s)}) = O(s^{\frac{5}{6}}e^{-m(s)}).$$

Similarly,

$$e^{-\phi_s(z)} - e^{-\nu_s(z)} = -e^{-p(z)}(\phi_s(z) - \nu_s(z))$$

for some $p(z)$ between $\phi_s(z)$ and $\nu_s(z)$. By considering the change of coordinate $w = s^{\frac{1}{k+3}}z$, we see for all $z \in \mathbb{C}$

$$\nu_s(z) = -\frac{1}{k+3}\log(s^2) + \nu_1(s^{-\frac{1}{k+3}}z).$$

Now ν_1 is bounded below, as in [DW15, Corollary 5.2], which implies

$$e^{-\nu_s(z)} = O(s^{\frac{2}{k+3}}).$$

Lemma 3.6 then shows

$$e^{-\phi_s(z)} - e^{-\nu_s(z)} = O(s^{\frac{2}{k+3}+\frac{1}{6}}e^{-m(s)}).$$

The result then follows if we show a decay estimate holds for $\partial_z(\phi_s - \nu_s)$ and $\partial_{\bar{z}}(\phi_s - \nu_s)$. Set $\eta_s = \phi_s - \nu_s$. We know from the computations in Lemma 3.6 that

$$\Delta_{h_s}\eta_s = 2(e^{\eta_s} - 1) - 4\frac{|q_s|^2}{e^{3\nu_s}}(e^{-2\eta_s} - 1).$$

Then $|q_s|^2/e^{3\nu_s}$ is uniformly bounded, because $|q_s|^2 = s^2|z|^{2k}$ and, using the same notation as in Lemma 3.5 and the subsolution for ψ_s found in [DW15, Theorem 5.1],

$$\begin{aligned}3\nu_s &= 3\psi_s + \frac{6}{k+3}\log(s)\\ &\geq \log(2s^{\frac{2k}{k+3}}|z|^{2k}) + \log(s^{\frac{6}{k+3}})\\ &= \log(2s^2|z|^{2k})\ .\end{aligned}$$

So

$$\|\Delta_{h_s}\eta_s\|_\infty \leq Cs^{\frac{1}{6}}e^{-m(s)}$$

and, using the supersolution for ν_s in [DW15, Theorem 5.1],

$$\|\partial_z\partial_{\bar z}\eta_s\|_\infty \leq \|e^{\nu_s}\|_\infty\|\Delta_{h_s}\eta_s\|_\infty \leq C(|q_s|^2+a)^{\frac{1}{3}}s^{\frac{1}{6}}e^{-m(s)} \leq Cs^{\frac{5}{6}}e^{-m(s)}.$$

The bounds on $\partial_z(\phi_s - \nu_s)$ and $\partial_{\bar z}(\phi_s - \nu_s)$ then follow from the Schauder and L^p estimates. □

The Stokes rays in B divide B into open *sectors*. We focus on *subsectors* Υ of these sectors, defined with opening angles strictly bounded away from those defining the sectors.

Proposition 4.2. *Let Υ be a subsector of B. For every $\gamma > 0$, there is $s_0 > 0$ so that for all $z \in \Upsilon$ and $s > s_0$,*

$$\|F_M(s^{\frac{1}{k+3}}z)\Theta_s F_M^{-1}(s^{\frac{1}{k+3}}z)\|_\infty \leq \gamma.$$

Proof. Let F_T denote the frame field of the standard Ţiţeica surface (see Example 2.1) with cubic differential dx^3 on the plane, which can be explicitly be written as

$$F_T(x) = S\ \exp\begin{pmatrix} 2^{\frac{2}{3}}\mathcal{R}e(x) & 0 & 0\\ 0 & 2^{\frac{2}{3}}\mathcal{R}e(x/\omega) & 0\\ 0 & 0 & 2^{\frac{2}{3}}\mathcal{R}e(x/\omega^2)\end{pmatrix} S^{-1} \tag{4.2}$$

for $\omega = e^{2\pi i/3}$ and

$$S = \begin{pmatrix}1 & 1 & 1\\ 1 & \omega & \omega^2\\ 1 & \omega^2 & \omega\end{pmatrix}\ .$$

From [DW15, Lemma 6.4], we know that, outside a compact set $K \subset \mathbb{C}$ and in an open subsector Υ uniformly bounded away from the Stokes lines, we have

$$F_M(\frac{3}{k+3}y^{\frac{k+3}{3}}) = (A + o(1))F_T(\frac{3}{k+3}y^{\frac{k+3}{3}}) \tag{4.3}$$

as $y \to \infty$ in the subsector. Here A is a constant nonsingular matrix that only depends on the subsector Υ. Here we use $\frac{3}{k+3}y^{\frac{k+3}{3}}$ because the result of [DW15] is stated in terms of natural coordinates for q_0, instead of the coordinate centered at the zero of q_0. Therefore,

$$\left\|F_M\left(\frac{3}{k+3}y^{\frac{k+3}{3}}\right)\right\|_\infty \leq C\left\|F_T\left(\frac{3}{k+3}y^{\frac{k+3}{3}}\right)\right\|_\infty \qquad \text{for all } y \in \Upsilon,$$

and a similar inequality holds for the inverses of the frame matrices. Thus, recalling that we set $w = s^{\frac{1}{k+3}}z$ (so that $w^k dw^3 = sz^k dz^3$) and substituting $s^{\frac{1}{k+3}}z$ in for $\frac{3}{k+3}y^{\frac{k+3}{3}}$ in the above expression, we see by (4.3)

$$\begin{aligned}\|F_M(s^{\frac{1}{k+3}}z)\|_\infty \|F_M^{-1}(s^{\frac{1}{k+3}}z)\|_\infty &\le C\|F_T(s^{\frac{1}{k+3}}z)\|_\infty \|\{(A+o(1))F_T(s^{\frac{1}{k+3}}z)\}^{-1}\|_\infty \\ &\le C\|F_T(s^{\frac{1}{k+3}}z)\|_\infty \|F_T^{-1}(s^{\frac{1}{k+3}}z)\|_\infty.\end{aligned}$$

Note the L^∞ norm of F_T is given by the largest eigenvalue in the frame F_T from (4.2), while the L^∞ norm of F_T^{-1} is given by the inverse of the smallest eigenvalue of F_T. Then by [DW15, Equation (21)] and the discussion immediately surrounding that equation, we conclude that

$$\|F_M(s^{\frac{1}{k+3}}z)\|_\infty \|F_M^{-1}(s^{\frac{1}{k+3}}z)\|_\infty \le Ce^{c(\theta)s^{\frac{1}{3}}\frac{3}{k+3}\epsilon^{\frac{k+3}{3}}},$$

where $c(\theta) \le 2^{\frac{2}{3}}\sqrt{3}$ and achieves this maximum value when θ corresponds to a Stokes direction. In particular, when $z \in \Upsilon$, there is $\alpha > 0$ such that $c(\theta) \le 2^{\frac{2}{3}}\sqrt{3} - \alpha$. In the definition 3.1 of $m(s)$, noting that $r_0 = \frac{3}{k+3}\epsilon^{\frac{k+3}{3}}$, we may choose both δ sufficiently close to 1 and s sufficiently large so that

$$s^{\frac{5}{3}}e^{c(\theta)s^{\frac{1}{3}}\frac{3}{k+3}\epsilon^{\frac{k+3}{3}}}e^{-m(s)} \le e^{-\beta s^{\frac{1}{3}}}$$

for some $\beta > 0$. Therefore, by Lemma 4.1,

$$\|F_M(s^{\frac{1}{k+3}}z)\Theta_s F_M^{-1}(s^{\frac{1}{k+3}}z)\|_\infty \le Ce^{-\beta s^{\frac{1}{3}}} \le \gamma$$

for s sufficiently large, independently of $z \in \Upsilon$. □

Corollary 4.3. *There exists $s_0 > 0$ such that for all $s > s_0$ and $z \in \Upsilon$, we have*

$$G_s(z) = \mathrm{Id} + o(\mathrm{Id}) \qquad \text{as} \quad \mathrm{s} \to +\infty.$$

Proof. Let $z = te^{i\theta} \in \Upsilon$. Then by Lemma B.2 in [DW15] applied to $B(t) = F_M(s^{\frac{1}{k+3}}te^{i\theta})\Theta_s(t)F_M^{-1}(s^{\frac{1}{k+3}}te^{i\theta})$ with $t \in [0,\epsilon]$ and Proposition 4.2, we have

$$\|G_s(te^{i\theta}) - \mathrm{Id}\|_\infty \le C\gamma,$$

which can be made arbitrarily small as $s \to +\infty$. □

In particular, this implies that for all $z \in \Upsilon$,

$$F_s(z) = G_s(z)F_M(s^{\frac{1}{k+3}}z) = (\mathrm{Id} + o(\mathrm{Id}))F_M(s^{\frac{1}{k+3}}z) \qquad \text{as} \quad s \to +\infty.$$

Combining this with the fact that $F_M(s^{\frac{1}{k+3}}z) = (A + o(\mathrm{Id}))F_T(s^{\frac{1}{k+3}}z)$, where A only depends on the sector in the complement of the Stokes line that contains z, we obtain

Corollary 4.4. *For every $z \in B$ inside the i^{th} sector in the complement of the Stokes rays,*

$$F_s(z) \cdot F_T^{-1}(s^{\frac{1}{k+3}}z) \overset{s\to+\infty}{\longrightarrow} A_i\ .$$

Here, of course, we have adapted our choice of Υ to our choice of z.

Corollary 4.5. *Let $I = [\theta_0, \theta_1]$ be a positively oriented circular arc containing one Stokes direction in its interior. Then*

$$A_0^{-1} A_1 = S U_I S^{-1},$$

where $U_I = U_{(\theta_0,\theta_1)}$ is one of the unipotents introduced in [DW15].

Proof. The matrices A_i are defined as the limit of $F_M(s^{\frac{1}{k+3}} e^{i\theta_i}) F_T^{-1}(s^{\frac{1}{k+3}} e^{i\theta_i})$ as $s \to \infty$. Thus the result follows from [DW15, Lemma 6.5]. □

Theorem 4.6 (Holonomy along arcs)**.** *Let $\gamma(t) = \epsilon e^{it}$ with $t \in I = [\theta_0, \theta_1]$ as in Corollary 4.5. Consider $G_t(s) = F_s(\gamma(t)) F_T^{-1}(s^{\frac{1}{k+3}} \gamma(t))$. Then*

$$\lim_{s \to +\infty} G_{\theta_0}^{-1}(s) G_{\theta_1}(s) = S U_I S^{-1},$$

where U_I is the same unipotent as in Corollary 4.5.

Proof. This follows immediately from Corollaries 4.4 and 4.5 because

$$\begin{aligned} G_{\theta_0}(s) &= F_s(\epsilon e^{i\theta_0}) F_T^{-1}(s^{\frac{1}{k+3}} \epsilon e^{i\theta_0}) \to A_0, \\ G_{\theta_1}(s) &= F_s(\epsilon e^{i\theta_1}) F_T^{-1}(s^{\frac{1}{k+3}} \epsilon e^{i\theta_1}) \to A_1, \end{aligned}$$

and $A_0^{-1} A_1 = S U_I S^{-1}$. □

We remark that when the interval $[\theta_0, \theta_1]$ contains more than one Stokes direction we can still apply Corollary 4.5 and Theorem 4.6 after splitting the interval $[\theta_0, \theta_1]$ into subintervals containing only one Stokes direction and thus satisfying the assumptions of Corollary 4.5. Because the holonomy is multiplicative along concatenation of paths, we will have

$$\lim_{s \to +\infty} G_{\theta_0}^{-1}(s) G_{\theta_1}(s) = S U S^{-1},$$

where U is now a *product* of unipotents depending on the Stokes directions the arc crosses. Of course, in this setting the product U is not necessarily a unipotent itself.

5. Asymptotic holonomy

We want to compute the asymptotic holonomy of the flat connection ∇^s on the rank-3 bundle $E = \mathbb{1} \oplus T_{\mathbb{C}}\Sigma$ along a $|q_0|^{\frac{2}{3}}$-geodesic path that may cross some of the zeros of the cubic differential q_0. This is the quotient of the bundle $\mathbb{1} \oplus T_{\mathbb{C}}\mathcal{D}$ introduced in Section 2 by the $\pi_1(\Sigma)$-action. We say such a geodesic path γ is *regular* if each segment away from the zeros of q_0 is

- not in the directions of the walls of a Weyl chamber, so that $\mathcal{R}e(\gamma^* \phi_i) \neq \mathcal{R}e(\gamma^* \phi_j)$ for $i \neq j$ (here ϕ_i is a root of $p(\lambda) = \lambda^3 - q_0$);
- not in the Stokes directions.

We later remove these hypotheses.

It is convenient to work in the universal cover of S. Equip S with the conformal hyperbolic metric and identify $\tilde{S}$ with the strip model of the hyperbolic plane

$$\mathbb{H}^2 = \{ w = \mu + i\nu \in \mathbb{C} : |\mu| < \pi/2 \}$$

with metric $g_{\mathbb{H}^2} = \frac{d\mu^2+d\nu^2}{\cos^2(\mu)}$. The vertical line $\mu = 0$ with arc-length parameter ν is a geodesic; so a hyperbolic deck transformation can be represented, up to conjugation, by the transformation $T(w) = w + iL$, where L is the translation length.

Remark 5.1. We want to use this model because it gives a way of defining a frame on E which we can use to compute parallel transport. The bundle E lifts to a bundle $\tilde{E}$ over $\mathbb{H}^2$ which we now trivialize using the global frame $\mathcal{F} = \{1, \partial_w, \partial_{\bar{w}}\}$. This frame is not parallel with respect to ∇^s. However, because it is globally defined, we can define the holonomy of $\tilde{\nabla}^s$ along an arc as a comparison between the terminal parallel transport of a frame in the fixed basis and the frame at the terminal point. Given $[\gamma] \in \pi_1(S)$, we can assume that the hyperbolic isometry corresponding to $[\gamma]$ is $T_\gamma(w) = w + iL$ for some $L \in \mathbb{R}$. Fix $w_0 \in \mathbb{H}^2$ and let $w_1 = T_\gamma(w_0)$. Note that $\mathcal{F}(w_0) = T_\gamma^*\mathcal{F}(w_1)$. Let $c_\gamma(t)$ be the $|q_0|^{\frac{2}{3}}$-geodesic connecting w_0 and w_1 with $t \in [0,1]$. If we have a matrix representation $M(c_\gamma(t))$ of the parallel transport along $c_\gamma(t)$ with respect to the frame $\mathcal{F}$, then the matrix $M(c_\gamma(1))$ represents the holonomy of the flat connection between the final and initial points as their frames are identified in the quotient.

Remark 5.2. We can assume that w_0 and w_1 are not zeros of q_0, so that the geodesic path c_γ starts and ends with a segment not containing any zeros.

The path c_γ will in general cross some of the zeros $p_1, \dots, p_\ell$ of the cubic differential q_0 with multiplicities $k_1, \dots, k_\ell$, respectively. In fact, we write c_γ as the union of $c_1, \dots, c_\ell$, where each c_i is the straight line path in the flat coordinates for $|q_0|^{\frac{2}{3}}$ from p_i to p_{i-1}. Each c_i does not intersect any zeros of q_0 except at its endpoints. We fix $\epsilon > 0$ and identify a neighborhood N_i of each zero and a conformal coordinate z so that $q_0 = z^{k_i} dz^3$ on N_i and the $|q_0|^{\frac{2}{3}}$-radius of N_i is ϵ. Note for ϵ small the closures $\overline{N_i}$ do not intersect. We modify c_γ to form a new path $\tilde{c}_\gamma$ by deleting each $c_\gamma \cap N_i$ and replacing it with an arc β_i in ∂N_i so that $\tilde{c}_\gamma$ is continuous and homotopic to the original geodesic. Now $c_\gamma \setminus \cup_i \overline{N_i}$ consists of a number of line segments $\tilde{c}_i \subset c_i$ in the flat q_0-coordinates. Divide each line segment $\tilde{c}_i$ between N_i and N_{i+1} into two segments $\tilde{\delta}_i$ and $\tilde{\alpha}_{i-1}$, so that each $\overline{N_i}$ has an incoming line segment $\tilde{\alpha}_i$ and an outgoing one $\tilde{\delta}_i$.

In total, $\tilde{c}_\gamma$ is the concatenation of $\tilde{\alpha}_\ell, \beta_\ell, \tilde{\delta}_\ell, \dots, \tilde{\alpha}_1, \beta_1, \tilde{\delta}_1$. The basepoint is $p = \tilde{\alpha}_\ell \cap \tilde{\delta}_1$. We also denote by α_i and δ_i the prolongments of $\tilde{\alpha}_i$ and $\tilde{\delta}_i$ to their forward and backward zero respectively. Since $\tilde{c}_\gamma$ and c_γ are homotopic, the holonomies along these paths are the same. Then

$$\mathrm{Hol}_s(\tilde{c}_\gamma) = \mathrm{Hol}_s(\tilde{\delta}_1)\mathrm{Hol}_s(\beta_1)\mathrm{Hol}_s(\tilde{\alpha}_1) \cdots \mathrm{Hol}_s(\tilde{\delta}_\ell)\mathrm{Hol}_s(\beta_\ell)\mathrm{Hol}_s(\tilde{\alpha}_\ell).$$

We want to find estimates for each factor and arrive at something of the form

$$\mathrm{Hol}_s(\tilde{c}_\gamma) = A(s) + E(s), \qquad \|E(s)\| = o(\|A(s)\|)$$

with $A(s)$ explicit and depending only on q_0 and on the geodesic path c_γ.

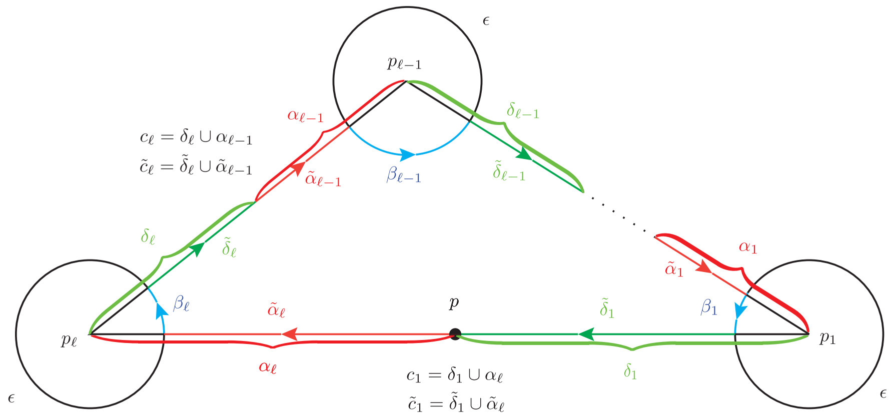

FIGURE 1. Definition of the subpaths.

Remark 5.3. Let $z = z(w)$ be a conformal change of coordinates. For instance, let z be a natural coordinate for q_s. The coordinate z induces a new frame $\mathcal{G} = \{1, \partial_z, \partial_{\bar{z}}\}$ of the bundle E. There is a diagonal matrix $d(z)$ depending on the derivatives of z so that $\mathcal{F} = \mathcal{G}\, d(z)$. Moreover, if z_1 and z_2 are two natural coordinates at a point, then z_1 and z_2 differ by a translation and a multiplication by a third root of unity. If we choose the natural coordinates so that they induce the same frame on the overlaps, we can multiply the matrices representing the parallel transport along consecutive arcs.

Remark 5.4. If $(\mathcal{U}, z_1)$ and $(\mathcal{V}, z_2)$ are two natural coordinate charts that cover a path γ and overlap at a point a, we note that z_1 and z_2 induce the same frame at a if and only if the path γ makes the same angle with the positive horizontal axis, as seen in the coordinates z_1 and z_2.

Let $a, a' \in \mathbb{H}^2$ and denote by $T_{a,a'}$ the parallel transport from a to a' for the lift $\tilde{\nabla}^s$ of the flat connection. Assume a and a' are not zeros of q_0 and are in the same natural coordinate z. Let $\mathcal{G}(a) = \{1, \partial_z, \partial_{\bar{z}}\}$ be the standard frame induced by z. The frame $\mathcal{G}$ is defined at a' as well; so we can find a matrix $\Psi_s(a')$ such that

$$T_{a,a'}(\mathcal{G}(a))\Psi_s(a') = \mathcal{G}(a').$$

Let $z(t)$ be a path connecting a and a'. The parallel transport condition is equivalent to $\Psi_s(z(t))$ being a solution of the initial value problem

$$\begin{cases} \Psi_s(z(0)) = \mathrm{Id}, \\ \Psi_s^{-1} d\Psi_s = U_s dz + V_s d\bar{z}, \end{cases}$$

where U_s and V_s are defined in Equation (4.1). The matrix representing the parallel transport $T_{a,a'} \colon \tilde{E}_a \to \tilde{E}_{a'}$ with respect to the frames $\mathcal{G}(a)$ and $\mathcal{G}(a')$ is then $\Psi_s(a')^{-1}$.

In what follows, instead of solving the initial value problem above, we compare Ψ_s with the solution Ψ_T of the initial value problem

$$\begin{cases} \Psi_T(z(0)) = \mathrm{Id}, \\ \Psi_T^{-1} d\Psi_T = U_T dz + V_T d\bar{z}, \end{cases}$$

where U_T and V_T are the matrices appearing in the structure equations for the affine sphere over $\mathbb{C}$ with constant cubic differential dz^3. We know ([Lof07]) that

$$\Psi_T(|w|e^{i\theta}) = S\exp(|z|D(\theta))S^{-1},$$

where

$$D(\theta) = \begin{pmatrix} 2^{\frac{2}{3}}\cos\theta & 0 & 0 \\ 0 & 2^{\frac{2}{3}}\cos(\theta - 2\pi/3) & 0 \\ 0 & 0 & 2^{\frac{2}{3}}\cos(\theta - 4\pi/3) \end{pmatrix}. \tag{5.1}$$

and S is the conjugating matrix that appeared e.g. in Proposition 4.2.

Remark 5.5. Note that Ψ_s solves the same ODE as the frame field F_s of the associated affine sphere. They differ by the value at the initial point.

The first author, in [Lof07], considered the case of geodesic paths which do not hit any zeros and determined the asymptotic behavior of the eigenvalues along such paths. We would like to use Proposition 3 of [Lof07], but unfortunately the published statement must be modified to Proposition 5.6 below, as there is a gap in the proof. The final paragraph of the proof in [Lof07] is unsupported. The main theorem of [Lof07] is still true, as follows from the results presented here. The main additional technique needed, which was available at the writing of [Lof07], is the fact that the largest eigenvalue of the holonomy along a path (and the reverse path) is enough to determine all the eigenvalues in $\mathrm{SL}(3,\mathbb{R})$. The first author regrets the error.

Thus we have the following proposition. We note that Collier-Li and Mochizuki have proved stronger estimates in a more general setting in the case in which no two eigenvalues are equal [CL17, Moc16].

Proposition 5.6 (Holonomy along rays)**.** *The parallel transports along the segments $\tilde{\alpha}_i$ and $\tilde{\delta}_i$ with respect to the frame $\mathcal{G}$ induced by a natural coordinate z for q_s are given by the matrices*

$$\begin{aligned} \mathrm{Hol}_s(\tilde{\alpha}_i) &= S\mathrm{diag}(e^{s^{\frac{1}{3}}\tilde{\mu}_1^i}, e^{s^{\frac{1}{3}}\tilde{\mu}_2^i}, e^{s^{\frac{1}{3}}\tilde{\mu}_3^i})S^{-1} + o(e^{s^{\frac{1}{3}}\tilde{\mu}^i}), \\ \mathrm{Hol}_s(\tilde{\delta}_i) &= S\mathrm{diag}(e^{s^{\frac{1}{3}}\tilde{\lambda}_1^i}, e^{s^{\frac{1}{3}}\tilde{\lambda}_2^i}, e^{s^{\frac{1}{3}}\tilde{\lambda}_3^i})S^{-1} + o(e^{s^{\frac{1}{3}}\tilde{\lambda}^i}), \end{aligned}$$

as $s \to +\infty$, where

$$\tilde{\mu}_j^i = -2^{\frac{2}{3}}\mathcal{R}e\left(\int_{\tilde{\alpha}_i}\phi_j\right), \qquad \tilde{\lambda}_j^i = -2^{\frac{2}{3}}\mathcal{R}e\left(\int_{\tilde{\delta}_i}\phi_j\right),$$

ϕ_j are the roots of $\lambda^3 - q_0$, so that $\phi_j = e^{\frac{2\pi\sqrt{-1}(j-1)}{3}}\phi_1$, and

$$\tilde{\mu}^i = \max\{\tilde{\mu}_1^i, \tilde{\mu}_2^i, \tilde{\mu}_3^i\}, \qquad \tilde{\lambda}^i = \max\{\tilde{\lambda}_1^i, \tilde{\lambda}_2^i, \tilde{\lambda}_3^i\}.$$

Remark 5.7. The error bounds in the previous proposition can be improved to $O(e^{s^{\frac{1}{3}}(\tilde{\mu}^i - C)})$ for C a positive constant depending the q_0-distance of the path to the zero set of q_0, by using Lemma 3.4 instead of the coarser bounds used in [Lof07].

Note that if we parametrize the path $\tilde{\delta}_i$ by $\tilde{\delta}_i(t) = te^{i\theta_i}$ with $t \in [\epsilon', L_i/2]$ for L_i the q_0-length of the geodesic segment between successive zeros of q_0 and $\epsilon' = \frac{3}{k+3}\epsilon^{\frac{k+3}{3}}$, then

$$\mathcal{R}e\left(\int_{\tilde{\delta}_i} \phi_j\right) = \mathcal{R}e\left(\int_{\epsilon'}^{L_i/2} \tilde{\delta}_i^* \phi_j\right) = (L_i/2 - \epsilon')\cos(\theta_i - 2(j-1)\pi/3).$$

Hence the position of the largest eigenvalue of the diagonal matrix depends only on the angle that $\tilde{\delta}_i$ makes with the positive x-axis in the chosen natural coordinates.

Remark 5.8. There is a relation between the diagonal matrices in Proposition 5.6 and Ψ_T. Precisely, if $w = re^{i\theta}$ and $\gamma(t) = te^{i\theta}$ with $t \in [0, r]$, then

$$\Psi_T(w) = S\mathrm{diag}(e^{-\mu_1}, e^{-\mu_2}, e^{-\mu_3})S^{-1} =: SD(\gamma)S^{-1},$$

where

$$\mu_j = -2^{\frac{2}{3}}\mathcal{R}e\left(\int_\gamma \phi_j\right),$$

and we choose the same conjugating matrix S as in Section 4.2.

Now, because the paths $\tilde{\delta}_i$ and $\tilde{\alpha}_i$ are part of a geodesic for the flat metric $|q_0|^{\frac{2}{3}}$, the angle between them is at least π (measured in the singular flat metric) at either side. Since Stokes rays are $\pi/3$ apart, the interval $[\theta_i, \theta_{i+1}]$ contains at least three Stokes directions.

We now compute the parallel transport along the circular arcs β_i in terms of the unipotents associated to the Stokes rays (cf. Corollary 4.5). In the z-coordinate centered at a zero so that $q_s = sz^{k_i}dz^3$, the path β_i is parametrized by $\beta_i(\theta) = \epsilon e^{i\theta}$ with $\theta \in [\theta_i, \theta_{i+1}]$. Let $w = \frac{3}{k+3}\omega^{j_i}s^{\frac{1}{3}}z^{\frac{k+3}{3}}$ be a natural coordinate for q_s. Choose $j_i \in \{1, 2, 3\}$ so that the angle the incoming path α_i makes with the positive x-axis coincides with the angle we saw in the previous natural coordinate chart.

Proposition 5.9 (Holonomy along arcs)**.** *Assume θ_i and θ_{i+1} do not correspond to Stokes directions. Then the holonomy along β_i satisfies*

$$\mathrm{Hol}_s(\beta_i) = F_T^{-1}(\beta_i(\theta_{i+1}))(SU(\theta_i, \theta_{i+1})^{-1}S^{-1} + o(\mathrm{Id}))F_T(\beta_i(\theta_i)) \qquad \text{as } s \to +\infty$$

with respect to the frame induced by the natural coordinate. Here $U(\theta_i, \theta_{i+1})$ is a product of unipotent matrices depending on which Stokes rays the path β_i crosses.

Proof. Recall that $\mathrm{Hol}_s(\beta_i)$ is the inverse of the matrix Ψ_s which solves the initial value problem

$$\begin{cases} \Psi_s^{-1}d\Psi_s = U_s dz + V_s d\bar{z}, \\ \Psi(\beta_i(\theta_i)) = \mathrm{Id}. \end{cases}$$

The frame field $F_s(z)$ is a solution of the same ODE with different initial conditions, so

$$\Psi_s(z) = F_s(\epsilon e^{i\theta_i})^{-1} F_s(z)$$

is the solution of the above initial value problem. Now, by Corollaries 4.4 and 4.5 and the subsequent remark,

$$\lim_{s\to+\infty} F_T(s^{\frac{1}{k+3}}\epsilon e^{i\theta_i})F_s^{-1}(\epsilon e^{i\theta_i})F_s(\epsilon e^{i\theta_{i+1}})F_T^{-1}(s^{\frac{1}{k+3}}\epsilon e^{i\theta_{i+1}}) = A_i^{-1}A_{i+1} = SUS^{-1},$$

where $U = U(\theta_i, \theta_{i+1})$ is as in the statement. Hence

$$\Psi_s(\epsilon e^{i\theta_{i+1}}) = F_s(\epsilon e^{i\theta_i})^{-1}F_s(\epsilon e^{i\theta_{i+1}}) = F_T^{-1}(s^{\frac{1}{k+3}}\epsilon e^{i\theta_i})(SUS^{-1}+o(1))F_T(s^{\frac{1}{k+3}}\epsilon e^{i\theta_{i+1}}).$$

Because $\mathrm{Hol}_s = \Psi_s^{-1}$ the claim follows. □

Combining the holonomy of each subpath (Proposition 5.9 and Proposition 5.6), we obtain

$$\begin{aligned}\mathrm{Hol}_s(\tilde{c}_\gamma) \quad = \quad & \prod_{i=1}^{\ell}\left(SD(\tilde{\delta}_i)^{-1}S^{-1} + o(e^{s^{\frac{1}{3}}\tilde{\lambda}^i})\right)\cdot \\ & SD(\delta_i \setminus \tilde{\delta}_i)^{-1}S^{-1}(SU(\theta_i,\theta_{i+1})^{-1}S^{-1} + o(\mathrm{Id}))SD(\alpha_i \setminus \tilde{\alpha}_i)^{-1}S^{-1}\cdot \\ & \left(SD(\tilde{\alpha}_i)^{-1}S^{-1} + o(e^{s^{\frac{1}{3}}\tilde{\mu}^i})\right)\end{aligned} \tag{5.2}$$

with respect to a frame induced by natural coordinates. Here $e^{s^{\frac{1}{3}}\tilde{\lambda}^i}$ and $e^{s^{\frac{1}{3}}\tilde{\mu}^i}$ are the largest eigenvalues of $D(\tilde{\delta}_i)^{-1}$ and $D(\tilde{\alpha}_i)^{-1}$ respectively.

Remark 5.10. The holonomy with respect to the global frame $\mathcal{F}$ will only differ by multiplication on the left and on the right by the change of frame between the global coordinate on $\mathbb{H}^2$ and the natural coordinate for q_s. These, however, only grow polynomially in s, so they do not influence the estimates that follow.

We can then write

$$\mathrm{Hol}(\tilde{c}_\gamma) = A(s) + E(s)$$

by setting

$$A(s) = \prod_{i=1}^{\ell} SD(\delta_i)^{-1}U(\theta_i,\theta_{i+1})^{-1}D(\alpha_i)^{-1}S^{-1}. \tag{5.3}$$

It is worth emphasizing that the definition of $A(s)$ does not depend on the local constructions around the zeros, and so applies both to the modified path $\tilde{c}_\gamma$ and to the original (regular) flat geodesic c_γ.

Lemma 5.11. *Along any regular geodesic, the following asymptotic expansion holds:*

$$A(s) = \prod_{i=1}^{\ell} c_{j_i,k_i} e^{s^{\frac{1}{3}}(\mu^i_{k_i}+\lambda^i_{j_i})} SE_{j_i,k_i}(\mathrm{Id}+o(\mathrm{Id}))S^{-1}, \tag{5.4}$$

where $E_{j,k}$ denotes the elementary matrix with 1 in position (j,k) and c_{j_i,k_i} is a non-zero constant. Here $\mu^i_{k_i} = \mu^i$ and $\lambda^i_{j_i} = \lambda^i$ are $s^{-\frac{1}{3}}$ times the logarithms of the largest eigenvalues of $D(\alpha_i)^{-1}$ and $D(\delta_i)^{-1}$ respectively.

Proof. This is a consequence of Proposition 6.3 below, whose precise statement we defer until later, as it depends on terminology developed in Section 6. This proposition shows that the element in position (j_i,k_i) of $U(\theta_i,\theta_{i+1})^{-1}$ is non-zero. We then factor each term in the triple product in Equation (5.3). □

Remark 5.12. Two consecutive terms in the above product have the property that $E_{j_i,k_i} \cdot E_{j_{i+1},k_{i+1}} = E_{j_i,k_{i+1}}$ (since $k_i = j_{i+1}$) because the position of the highest eigenvalue only depends on the angle the path makes with the x-axis in a natural coordinate and our choices of coordinates keep this angle constant when the coordinate patches cover the same straight path. Note, indeed, that α_i and δ_{i+1} (with indices intended modulo ℓ) are part of the same straight line segment c_{i+1}.

We also give an argument to address the special case in which the the angle of a geodesic segment is in a Stokes direction, extending Lemma 5.11 to the case when Proposition 5.9 does not hold. Define $\tilde{c}_i = \tilde{\delta}_i \cup \tilde{\alpha}_{i-1}$ to be the geodesic segment in $\tilde{c}_\gamma$ corresponding to this geodesic arc. Then the estimates of [DW15] fail to hold at its endpoints $\tilde{c}_i \cap \beta_i$ and $\tilde{c}_i \cap \beta_{i-1}$. We will modify $\tilde{c}_i, \beta_i, \beta_{i-1}$ slightly by moving the endpoints. Recall $c_i = \delta_i \cup \alpha_{i-1}$ is the corresponding geodesic segment between the zeros in γ.

We rewrite (5.3) as

$$A(s) = SD(\alpha_\ell)\left[\prod_{i=1}^{\ell} D(c_i)^{-1}U(\theta_i,\theta_{i+1})^{-1}\right] D(\alpha_\ell)^{-1}S^{-1}. \tag{5.5}$$

Proposition 5.13. *Lemma 5.11 holds for homotopy classes of free loops for which the flat geodesic's saddle connection segments are all either regular or travel along Stokes rays: in other words, if no saddle connection is contained in a wall of a Weyl chamber.*

Proof. It suffices to address the case of a single saddle connection along a Stokes ray. Each endpoint of the $\tilde{c}_i$ is in the flat coordinate $\epsilon e^{i\theta}$ for θ a Stokes direction. We modify the angle by $\pm\eta$ for a small positive constant η to avoid these directions. So define β_i^η to be the new arc formed by replacing the endpoint $\epsilon e^{i\theta}$ by $\epsilon e^{i(\theta\pm\eta)}$, and similarly define β_{i-1}^η. Define $\tilde{c}_i^\eta$ to be the geodesic path between these endpoints of β_i^η and β_{i-1}^η. By choosing η small enough we can ensure the straight line homotopy between $\tilde{c}_i$ and $\tilde{c}_i^\eta$ does not cross any other zeros of the cubic differential.

In certain cases we also need to specify the signs $\pm\eta$. At each zero along the geodesic γ, the incoming and outgoing rays must make an angle of $\geq \pi$ with respect the flat metric, when measured in clockwise and counterclockwise directions around the zero. Proposition 6.3 below requires that each arc begins and ends away from a Stokes ray and must subtend an angle $> \pi$ (and so at least 3 Stokes rays will be transversed by the arc). For each endpoint of $\tilde{c}_i$ choose $\pm\eta$ so that the arcs β_i^η and

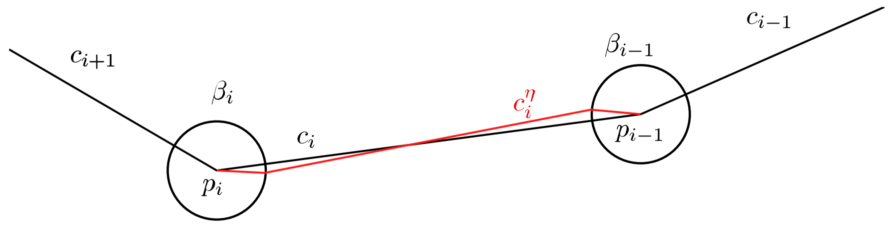

FIGURE 2. Path modified by η.

β^η_{i-1} both subtend an angle $> \pi$. This is possible since the total angle around a zero of order k is $2\pi + 2\pi k/3$. Note that in some cases we are free to choose either $+\eta$ or $-\eta$; then there is a different holonomy matrix along the arc depending on the sign. Lemma 6.5 below shows that this matrix leaves the relevant entries unchanged, in terms of the leading order terms.

Define c_i^η to be the union of $\tilde{c}_i^\eta$ and the two radial paths from the zeros p_{i-1}, p_i to the endpoints of $\tilde{c}_i^\eta$. See Figure 2. Now by Proposition 5.6 and Remark 5.8, the contribution for c_i^η in (5.5) is given by

$$S\mathrm{diag}(e^{s^{\frac13}\nu_1^{i,\eta}}, e^{s^{\frac13}\nu_2^{i,\eta}}, e^{s^{\frac13}\nu_3^{i,\eta}})S^{-1} + o(e^{s^{\frac13}\nu^{i,\eta}})$$

where

$$\nu_j^{i,\eta} = -2^{\frac23}\mathcal{R}e\left(\int_{c_i^\eta}\phi_j\right) = -2^{\frac23}\mathcal{R}e\left(\int_{c_i}\phi_j\right) = \nu_j^i = \mu_j^{i-1} + \lambda_j^i,$$

since ϕ_j is closed and c_i^η is homotopic to c_i. This shows as above that the conclusion of Lemma 5.11 holds.

Note we call the entire modified path $\tilde{c}^\eta$. It is obtained from $\tilde{c}$ by replacing, for each appropriate i, $\tilde{c}_i$ by $\tilde{c}_i^\eta$ and β_i by β_i^η, etc. □

6. NO BRANCHING

In this section we exploit the geometry of the convex regular polygon to which a hyperbolic affine sphere with cubic differential $z^k dz^3$ projects in order to analyze the non-zero entries of the product of unipotent matrices $U(\theta_i, \theta_{i+1})^{-1}$ of the previous section. The key result, Proposition 6.3, asserts that the product of unipotents in the holonomy formula (5.3) – that connect the holonomy of segments that come into a zero with the holonomy of segments that leave a zero – have an entry that allows the largest eigenvalues of those holonomies to multiply. This is crucial for the form of the formula (5.3).

Proposition 6.3 below will be used later to show that the induced map from the Riemann surface to the real building given by the asymptotic cone is locally injective near the zeros of the cubic differential, and thus can have no branching behavior. The corresponding phenomenon in the real tree case is called "folding," which does occur in some situations (e.g. [DDW00], [Wol07]).

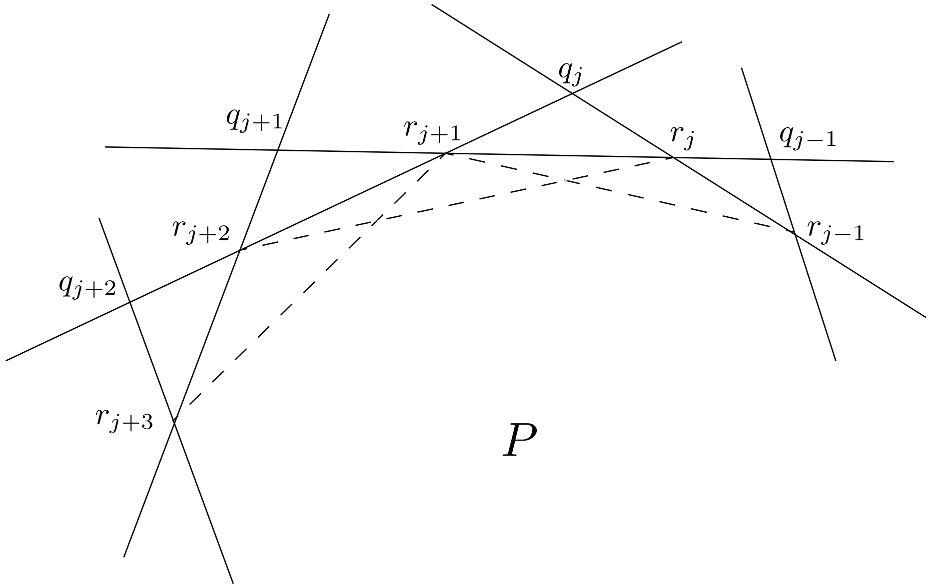

FIGURE 3. Inscribed and circumscribed triangles. Here each inscribed triangle includes a dotted edge and each circumscribed triangle contains one edge of the polygon and extends the immediate neighbors of that edge to meet a point q_j exterior to the polygon.

Choose a local coordinate w on $\mathbb{C} \setminus \{0\}$ so that $q = dw^3$. Consider f the corresponding embedding of the Riemann surface into $\mathbb{R}^3$ whose image is a hyperbolic affine sphere with Pick differential q, and consider the frame $F = (f, f_w, f_{\bar{w}})$ of the affine sphere. Let f_T, F_T be the corresponding embedding and frame for a standard Ţiţeica surface (as described in Example 2.1). The osculation map $G(z) = F(z)F_T^{-1}(z)$ then has limits $SL_- S^{-1}, SL_0 S^{-1}, SL_+ S^{-1}$ along rays $\gamma(t) = te^{i\theta}$ of angle θ for $\theta \in (-\pi/2, -\pi/6)$, $(-\pi/6, \pi/6), (\pi/6, \pi/2)$ respectively ([DW15]). These matrices determine the construction of the convex polygon P onto which the affine sphere $f(\mathbb{C})$ projects. Let us summarize the main step of the construction. We label the vertices of P as $r_0, r_1, \ldots, r_{n-1} \in \mathbb{RP}^2$ with indices in $\mathbb{Z}_n$. Let $\overline{r_j r_{j+1}}$ denote the line connecting r_j and r_{j+1}, and let e_j denote the edge of P from r_j to r_{j+1}. Three successive vertices r_{j-1}, r_j, r_{j+1} form an inscribed triangle in P around the vertex r_j. Also define points $q_j = \overline{r_{j-1} r_j} \cap \overline{r_{j+1} r_{j+2}}$ so that r_j, q_j, r_{j+1} are the vertices of a circumscribed triangle T_j of P centered around the edge e_j. See Figure 3.

Proposition 6.1. *In the above setting the following holds:*

(1) $r_j \in \overline{r_i r_{i+1}}$ *if and only if* $j = i$ *or* $j = i + 1$.
(2) $q_j \in \overline{r_i r_{i+1}}$ *if and only if* $j = i - 1$ *or* $j = i + 1$.

In the case of P being a quadrilateral, (i.e. $2n = 8$ triangles and degree $k = 1$ of the cubic differential), the formulae remain valid, though we note that $q_0 = q_2$ and $q_{-1} = q_1 = q_3$.

Proof. Statement (1) is obvious from the convexity of P.

To prove statement (2), note we need only prove the "only if" part. So assume $q_j \in \overline{r_i r_{i+1}}$. As q_i, r_i, r_{i+1} are the vertices of a triangle, we see $j \neq i$. To find a contradiction, assume $j < i-1$ or $j > i+1$ and that $q_j \in \overline{r_i r_{i+1}}$. Recall $q_j = \overline{r_{j-1} r_j} \cap \overline{r_{j+1} r_{j+2}}$. By convexity, $T_j \supset P$. Since P is a convex polygon, it is the intersection of n closed half-planes H_k, each bounded by $\overline{r_k r_{k+1}} \supset e_k$. Since $q_j \in \overline{r_i r_{i+1}}$, we see P is a subset of the smaller triangle $T_j \cap H_i$, which cannot contain both e_{j-1} and e_{j+1}. This is a contradiction. □

The columns of the matrices L_-, L_+ project to the vertices of a circumscribed triangle of P, whereas the columns of L_0 project to the vertices of an inscribed triangle. Moreover, the middle vertex of an inscribed triangle is always obtained as $L_0 e_i$ where Se_i is the eigenvector corresponding to the highest eigenvalue of $F_T(z)$. The unipotent matrices arise then by taking the products $L_-^{-1} L_0$ and similar. We determine these unipotent matrices explicitly in the case P is regular. Choose a coordinate frame in $\mathbb{R}^3$ in which r_j takes the form

$$r_j = \left[\cos\frac{2\pi j}{n}, \sin\frac{2\pi j}{n}, 1\right]^t.$$

We then compute that

$$q_0 = \left[\frac{1}{2} + \frac{1}{2}\sec\frac{2\pi}{n}, \frac{1}{2}\tan\frac{2\pi}{n}, 1\right]^t,$$

while q_j can be computed as $r_j q_0$, where we view $r_j, q_0 \in \mathbb{C}$. In other words, in the natural inhomogeneous coordinate chart $\{[x, y, 1]^t\}$ in $\mathbb{RP}^2$, q_j is found by rotating q_0 by an angle of $2\pi j/n$. (Note that the formulas respect the circular indexing, i.e. $r_{-1} = r_{n-1}$. Also, the case of quadrilateral P (i.e. $n = 4$) requires special treatment here as we cannot locate r_j and q_j within the single inhomogeneous chart; here we set $q_0 = [1, 1, 0]^t = q_2$ and $q_1 = [1, -1, 0]^t = q_3$.)

We continue towards our goal (cf. Lemma 5.11, equation (5.3) and Figure 1) of understanding the effect of multiplying the unipotents that are encountered when a path around a zero crosses Stokes lines (as well as the changes of order of eigenvalues for the standard Ţiţeica comparison frame F_T as the path for the holonomy crosses walls of Weyl chambers). In particular, we are trying to compute the largest term of the asymptotic holonomy along our geodesic as we pass from from one saddle connection c_i to the next c_{i-1} along an arc β_{i-1} around a zero. Our aim in this section is to ensure that the largest terms in the holonomy matrix along c_i and c_{i-1} multiply via a positive term in the product of unipotents along β_{i-1}.

Now, the paper [DW15] provides a scheme of determining the projective transformations of triangles (across Stokes lines) to form the polygon, as well as the order of the largest eigenvalue corresponding to each vertex. See the tables on pages 1771-1772 in [DW15].

The diagram below organizes the information we need in three rows. We will describe this in detail, but here is an initial guide to the arrangement: the first row is meant to track the vertices of the triangle for the best approximating Ţiţeica surface for the sector, the second row tracks the relative magnitude of the eigenvalues for the approximating Ţiţeica frame F_T against the listed order of the vertices in the first row, while the third row tracks the angles in a standard flat coordinate.

$$
\begin{array}{ccccccccccccc}
\multicolumn{13}{c}{\mapsto \begin{pmatrix} r_{-1} \\ r_0 \\ r_1 \end{pmatrix} \mapsto \begin{pmatrix} q_0 \\ r_0 \\ r_1 \end{pmatrix} \mapsto \begin{pmatrix} r_2 \\ r_0 \\ r_1 \end{pmatrix} \mapsto \begin{pmatrix} r_2 \\ q_1 \\ r_1 \end{pmatrix} \mapsto \begin{pmatrix} r_2 \\ r_3 \\ r_1 \end{pmatrix} \mapsto \begin{pmatrix} r_2 \\ r_3 \\ q_2 \end{pmatrix} \mapsto} \\
\\
\begin{matrix} m \\ \ell \\ s \end{matrix} & \Bigg| & \begin{matrix} s \\ \ell \\ m \end{matrix} & \Bigg| & \begin{matrix} s \\ m \\ \ell \end{matrix} & \Bigg| & \begin{matrix} m \\ s \\ \ell \end{matrix} & \Bigg| & \begin{matrix} \ell \\ s \\ m \end{matrix} & \Bigg| & \begin{matrix} \ell \\ m \\ s \end{matrix} & \Bigg| & \begin{matrix} m \\ \ell \\ s \end{matrix} \\
\\
-\frac{\pi}{6} & 0 & \frac{\pi}{6} & \frac{\pi}{3} & \frac{\pi}{2} & \frac{2\pi}{3} & \frac{5\pi}{6} & \pi & \frac{7\pi}{6} & \frac{4\pi}{3} & \frac{3\pi}{2} & \frac{5\pi}{3} & \frac{11\pi}{6}
\end{array}
\tag{6.1}
$$

Here each $\mapsto$ in the first row refers to crossing a Stokes line, which in standard flat coordinates are at angles $-\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \dots$ as noted in the third row, while each vertical line in the second row refers to crossing a wall of a Weyl chamber, at angles $0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \dots$ (in the third row). The s, m, ℓ in the second row refer to the smallest, medium and largest eigenvalue respectively of the frame $F_T(z)$.

For example, in the angular sector $0 < \theta < \frac{\pi}{6}$, measured in a natural coordinate, the affine sphere over P is asymptotic to a unique Ţiţeica surface which projects onto the inscribed triangle T whose vertices are represented by the first vector (r_{-1}, r_0, r_1) in the diagram. The vertices of the triangle are the projective classes of the eigenspaces of $F_T(z)$. We can say more: the eigenvalues appear in the order (s, ℓ, m) relative to the ordering (r_{-1}, r_0, r_1) of the vertices. When we cross the Stokes line at $\theta = \pi/6$, the Ţiţeica surface best approximating the affine sphere over P flips and projects to the circumscribed triangle T' of vertices (q_0, r_0, r_1). However, the eigenvalues of $F_{T'}(z)$ still appear in the same ordering (s, ℓ, m). This changes to (s, m, ℓ) only after crossing the wall of the Weyl chamber at the angle $\pi/3$. At the wall the largest two eigenvalues coincide, but upon crossing the wall, the largest and middle eigenvalues swap. This crossing, though, leaves the triangle T', and hence the triple (q_0, r_0, r_1) invariant.

There is of course more than one projective transformation which takes a triangle to a triangle, but the ones we are interested in are determined by the conditions that they are unipotent and that they fix the vector lifts to $\mathbb{R}^3$ of the relevant r_j's fixed by the projective transformation. The next proposition shows, at least in the regular case, that these vector lifts of r_j, q_j can be done globally, and that the relevant linear transformations corresponding to crossing several Stokes lines near a zero can be computed in terms of simple changes of bases (in contrast to a more complicated scheme of multiplying out several transformations).

Proposition 6.2. *Let P be a regular convex polygon as above. There are vector lifts $\vec{r}_j, \vec{q}_j \in \mathbb{R}^3$ of the points $r_j, q_j \in \mathbb{RP}^2$ for which the linear transformations lifting the* *corresponding triangle transformations are all unipotent. The choice of such a set of lifts is unique up to a nonzero multiplicative constant.*

Proof. We may easily check that

$$\vec{r}_j = \left(\cos\frac{2\pi j}{n}, \sin\frac{2\pi j}{n}, 1\right)^t \quad \text{and} \quad \vec{q}_0 = \left(-\cos\frac{2\pi}{n} - 1, -\sin\frac{2\pi}{n}, -2\cos\frac{2\pi}{n}\right)^t$$

satisfy the conditions, with $\vec{q}_j$ again being defined by rotating $\vec{q}_0$ by an angle of $2\pi j/n$. □

(Again, we add a comment for the case of P a quadrilateral ($n = 4$): while $q_0 = q_2$ and $q_1 = q_3$, the lifts satisfy $\vec{q}_0 = -\vec{q}_2$ and $\vec{q}_1 = -\vec{q}_3$.)

It follows that the matrix $U(\theta_i, \theta_{i+1})^{-1}$ represents the change of basis between $\mathcal{B}_i = \{v_1, v_2, v_3\}$ and $\mathcal{B}_{i+1} = \{w_1, w_2, w_3\}$ where v_j and w_j project to the vertices of inscribed or circumscribed triangles of the regular polygon P. Therefore, if v_{k_i} is the vector corresponding to the highest eigenvalue of $D(\alpha_i)^{-1}$ and w_{j_i} is the vector corresponding to the highest eigenvalue of $D(\delta_i)^{-1}$, the entry (j_i, k_i) of $U(\theta_i, \theta_{i+1})^{-1}$ is nonzero if and only if v_{k_i} has a component along w_{j_i} in the basis $\mathcal{B}_{i+1}$. It is important to note, because of the orientations of the paths near a given zero p_i, that the highest eigenvalue of $D(\alpha_i)^{-1}$ is the highest eigenvalue of F_T, while the highest eigenvalue of $D(\delta_i)^{-1}$ is the lowest eigenvalue of F_T.

As promised above in Lemma 5.11, we now prove

Proposition 6.3. *Consider any convex polygon P. If the highest eigenvalue of $D(\delta_i)^{-1}$ is in position j_i and the highest eigenvalue of $D(\alpha_i)^{-1}$ is in position k_i, then the (j_i, k_i)-entry of $U(\theta_i, \theta_{i+1})^{-1}$ is not zero.*

Proof. Refer to Figure 3 and Equation 6.1. We recall the setting we discussed before Proposition 5.9: because the paths δ_i and α_i are part of a geodesic for the flat metric $|q_0|^{\frac{2}{3}}$, the angle between δ_i and α_i is at least π (measured in the singular flat metric) at either side. Since Stokes rays are $\pi/3$ apart, the interval $[\theta_i, \theta_{i+1}]$ contains at least three Stokes directions. Crossing a Stokes direction corresponds to "flipping" the initial triangle (i.e., moving to the next triangle in the sequence displayed in (6.1), accomplished geometrically between triangles that share an edge). In the rest of the proof, because it is only necessary to consider flips of at most n triangles – amounting to traversing halfway around the polygon – for the clarity of the exposition, all of our subsequent discussion will assume this restriction.

We assume that the initial basis $\{v_1, v_2, v_3\}$ projects to a circumscribed triangle and thus is of the form $\{\vec{r}_j, \vec{q}_j, \vec{r}_{j+1}\}$. In this case the eigenvector corresponding to the highest eigenvalue can project to either r_j or r_{j+1}: it projects to r_j if the direction θ_i is in the first half of the interval determined by two Stokes directions and to r_{j+1} otherwise. We will explain the argument in detail when the highest eigenvalue is in the direction of $\vec{r}_j$; the case of $\vec{r}_{j+1}$ or when the initial basis projects to an inscribed triangle are analogous and left to the reader. If the direction of the highest eigenvalue is $\vec{r}_j$, then after at least three flips in the counter-clockwise direction, by Proposition

6.1, the point r_j does not lie in any of the lines generated by the final triangle. Hence all entries in the first column of $U(\theta_i,\theta_{i+1})^{-1}$ are non-zero. If we move in clockwise direction instead, then after at least five flips the point r_i never lies on any of the lines generated by the vertices of the final triangle and the claim follows as before. Thus, we only need to check what happens when only three or four Stokes directions are contained in the interval $[\theta_i,\theta_{i+1}]$.

If the final triangle is obtained after four flips in the clockwise direction, then it is circumscribed and the vertex corresponding to the highest eigenvalue of $D(\delta_i)^{-1}$ is q_{j-2}, i.e. the one vertex outside the polygon. This is because, referencing Equation 6.1 again, we see both that the highest eigenvalue of $D(\delta_i)^{-1}$ is the lowest eigenvalue of $F_T(z)$ and that the corresponding eigenvector always projects to the vertex exterior to the polygon. Because $r_j \in \overline{r_{j-1}q_{j-2}}$ (see for example Figure 3), the coordinates of $\vec{r}_j$ with respect the final basis $\{\vec{r}_{j-1},\vec{q}_{j-2},\vec{r}_{j-2}\}$ have a nonzero component along $\vec{q}_{j-2}$, hence the corresponding entry in $U(\theta_i,\theta_{i+1})^{-1}$ is nonzero.

Finally, if the final basis is obtained after only three flips, it is easy to see that the highest eigenvalues of $D(\alpha_{i+1})^{-1}$ and $D(\delta_i)^{-1}$ are always in the same position: this follows again by a careful reading of Equation (6.1). (We add some details. Recall that we chose the initial point to be in the first half of the region between Stokes lines, i.e. in the region between the Stokes line and the Weyl chamber. For example, we are focused on a point in the region between $\frac{3\pi}{2}$ and $\frac{5\pi}{3}$; after three flips [which we read from right to left in Equation (6.1)], this point lands in the region between $\frac{\pi}{2}$ and $\frac{2\pi}{3}$. By inspection of the corresponding bases in (6.1), we see that the initial and terminal basis element, r_2, corresponding to the highest eigenvalue is unchanged, so the corresponding entry in the product of unipotents is non-vanishing.) The chain in Equation 6.1 shows that the triple (r_j,r_{j+1},q_j) is sent to (r_j,r_{j-2},r_{j-1}), so the elements on the diagonal are all non-zero.

Finally, we note that in the argument above, all the conditions checked involve only incidences of the given points and lines in $\mathbb{RP}^2$, and not the choices of vector lifts in $\mathbb{R}^3$. Thus the proposition holds not only for regular polygons, but for all convex polygons. □

Below in Proposition 6.8, we extend Proposition 6.3 to determine the signs of all the relevant entries. These signs will be useful in handling the special cases in which the angle of a geodesic segment in the flat metric is at a Stokes line or wall of a Weyl chamber.

We recall Cramer's Rule from linear algebra:

Lemma 6.4. *If v is a vector in $\mathbb{R}^m$, and $w_1,\dots,w_m$ is a basis, then $v=\sum_\alpha \lambda^\alpha w_\alpha$, where*

$$\lambda^\alpha = \frac{\det(w_1,\dots,w_{\alpha-1},v,w_{\alpha+1},\dots,w_m)}{\det(w_1,\dots,w_m)} =: \frac{\mathcal{D}_v}{\mathcal{D}}.$$

Assume for now that the polygon is regular, so that all the r_j,q_j lift to vectors $\vec{r}_j,\vec{q}_j$ as in Proposition 6.2. It is useful to set up some notation for the following results. We let $v=\vec{r}_k$ be the eigenvector with the largest eigenvalue of $D(\tilde{\alpha}_i)^{-1}$ for the incoming ray $\tilde{\alpha}_i$. The frame for the outgoing ray is $\{w_\alpha\}$, and $w \in \{w_\alpha\}$ is the one with the largest eigenvalue for $D(\tilde{\delta}_i)^{-1}$. We can write $\{w_\alpha\} \setminus \{w\} = \{\vec{r}_j, \vec{r}_{j+1}\}$ for some j.

Lemma 6.5. *$\mathcal{D}_v$ is unchanged if the incoming or outgoing angles vary by crossing a single Stokes line.*

Proof. Refer to equation (6.1). If the incoming angle moves across a single Stokes ray, the largest eigenvector v of F_T is unchanged: recall that in equation (6.1), each Stokes ray, denoted by $\mapsto$, bisects a Weyl chamber region, whose boundaries are denoted by vertical lines. If the outgoing angle is changed by crossing a single Stokes line, then the smallest eigenvector w of F_T is the only vector to change in the frame $\{w_\alpha\}$. Upon replacing w with v, we see $\mathcal{D}_v$ (and indeed the underlying matrix) is unchanged. □

Lemma 6.6. *$\mathcal{D}_v = \det(\vec{r}_j, \vec{r}_{j+1}, \vec{r}_k)$ for some $j, k \in \mathbb{Z}/n$. $\mathcal{D}_v > 0$ if $k \neq j, j+1$, and is 0 otherwise.*

Proof. We see v, as the eigenvector of the largest eigenvalue for an incoming ray, is of the form $v = \vec{r}_k$ for some k. Similarly, after possibly applying Lemma 6.5 and replacing $w \in \{w_\alpha\}$ with v, we see from 6.1, upon perhaps performing a cyclic permutation, that $\mathcal{D}_v = \det(\vec{r}_j, \vec{r}_{j+1}, \vec{r}_k)$ for some j.

If $k = j, j+1$, it is obvious that $\mathcal{D}_v = 0$. On the other hand, if $k \neq j, j+1$, then r_j, r_{j+1}, r_k form a counterclockwise-oriented triangle in the plane. By Proposition 6.2, $\vec{r}_j = (r_j, 1) \in \mathbb{R}^3$. So this orientation of the triangle implies $\mathcal{D}_v = \det(\vec{r}_j, \vec{r}_{j+1}, \vec{r}_k) > 0$. □

Lemma 6.7. $\mathcal{D} > 0$.

Proof. In the case of an inscribed triangle, $\mathcal{D} = \det(\vec{r}_j, \vec{r}_{j+1}, \vec{r}_{j+2})$ for some j. Thus as in the previous lemma, $\mathcal{D} > 0$. To see the statement for a circumscribed triangle, note that it holds for an inscribed triangle and then to pass from an inscribed to the subsequent circumscribed triangle, the frame is changed by a unipotent transformation, which leaves the determinant $\mathcal{D}$ unchanged. □

Proposition 6.8. *Let P be any convex polygon. Given the notation of Proposition 6.3, the (j, k) entry of $U(\theta_i, \theta_{i+1})^{-1}$ is positive.*

Proof. Proposition 6.3 shows the relevant entry is nonzero. For the particular case of regular polygons, Lemmas 6.4, 6.6, and 6.7 show that this entry is positive. Then the general result follows by continuity and the connectedness of the moduli space of convex n-gons. □

7. Main Theorem - asymptotics of singular values

In this section, we prove the asymptotics of singular values of the holonomy associated to geodesic paths. The singular values naturally give the distance in the symmetric space $\mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3)$. First, in Theorem 7.1, we focus on the case of regular geodesic paths (allowing Stokes directions). In these cases, we have shown there is always a unique largest element in each diagonal matrix $D(c_i)^{-1}$ in (5.5), while the relevant entries linking them together in $U(\theta_i, \theta_{i+1})^{-1}$ are positive. Later, in Theorem 7.5, we remove the remaining restrictions to allow any geodesic path without any restriction on the angles of the segments. Finally in Corollary 7.6 we extend the analysis beyond just singular values to include eigenvalues.

Theorem 7.1. *Using the same notation as in Lemma 5.11, for every regular path $\tilde{c}_\gamma$, and indeed for every path none of whose segments is contained in a wall of a Weyl chamber, we have*

$$\lim_{s\to+\infty} \frac{\log \|\mathrm{Hol}_s(\tilde{c}_\gamma)\|}{s^{\frac{1}{3}}} = \sum_{i=1}^{\ell} \nu^i,$$

where $\|\cdot\|$ is any submultiplicative matrix norm and ν^i is the largest of the $\nu^i_j = -2^{\frac{2}{3}}\mathcal{R}e\int_{c_i}\phi_j$.

Proof. Recall from Section 5 that $\mathrm{Hol}_s(\tilde{c}_\gamma) = A(s) + E(s)$. Assume for now that $\|E(s)\| = o(\|A(s)\|)$; we return to check this assumption after using it to conclude the theorem. Then, by the estimates of $A(s)$ in Lemma 5.11,

$$\begin{aligned}
\log \|\mathrm{Hol}_s(\tilde{c}_\gamma)\| &= \log \|A(s) + E(s)\| \\
&= \log \|A(s)\| + \log \left\| \frac{A(s)}{\|A(s)\|} + \frac{E(s)}{\|A(s)\|} \right\| \\
&= \log \prod_{i=1}^{\ell} e^{s^{\frac{1}{3}}(\mu^i_{k_i} + \lambda^i_{j_i})} \left\| \prod_{i=1}^{\ell} c_{j_i,k_i} S E_{j_i,k_i} (\mathrm{Id} + o(\mathrm{Id})) S^{-1} \right\| \\
&\quad + \log \left\| \frac{A(s)}{\|A(s)\|} + \frac{E(s)}{\|A(s)\|} \right\|,
\end{aligned}$$

where E_{j_i,k_i} denotes the (j_i, k_i) elementary matrix. Now,

$$\frac{A(s)}{\|A(s)\|} + \frac{E(s)}{\|A(s)\|} \to N, \qquad \text{with } \|N\| = 1, \text{ as } \frac{E(s)}{\|A(s)\|} \to 0$$

and

$$\prod_{i=1}^{\ell} c_{j_i,k_i} S E_{j_i,k_i} (\mathrm{Id} + o(\mathrm{Id})) S^{-1} = M + o(\mathrm{Id}), \quad M = S \left(\prod_{i=1}^{\ell} c_{j_i,k_i} E_{j_i,k_i} \right) S^{-1} \neq 0.$$

Hence,

$$\begin{aligned}\lim_{s\to+\infty}\frac{\log\|\mathrm{Hol}_s(\tilde{c}_\gamma)\|}{s^{\frac{1}{3}}} &= \lim_{s\to+\infty}\Bigg[\left(\sum_{i=1}^{\ell}\mu^i_{k_i}+\lambda^i_{j_i}\right)+\frac{\log\|M+o(\mathrm{Id})\|}{s^{\frac{1}{3}}} \\ &\quad + \frac{1}{s^{\frac{1}{3}}}\log\left\|\frac{A(s)}{\|A(s)\|}+\frac{E(s)}{\|A(s)\|}\right\|\Bigg] \\ &= \sum_{i=1}^{\ell}\mu^i_{k_i}+\lambda^i_{j_i}=\sum_{i=1}^{\ell}\mu^{i-1}_{k_{i-1}}+\lambda^i_{j_i}=\sum_{i=1}^{\ell}\nu^i,\end{aligned}$$

where as usual $i-1$ is considered modulo ℓ.

We only need to check the condition on the error: $\|E(s)\|=o(\|A(s)\|)$. For this estimate we are going to use the L^∞-norm $\|\cdot\|_\infty$. Note, however, that because all matrix norms are equivalent the result holds for any matrix norm. Recall that $E(s)$ is equal to

$$\begin{aligned}&\prod_{i=1}^{\ell}\left(SD(\tilde{\delta}_i)^{-1}S^{-1}+o(e^{s^{\frac{1}{3}}\tilde{\lambda}^i})\right)\cdot \\ &\qquad SD(\delta_i\setminus\tilde{\delta}_i)^{-1}S^{-1}(SU(\theta_i,\theta_{i+1})^{-1}S^{-1}+o(\mathrm{Id}))SD(\alpha_i\setminus\tilde{\alpha}_i)^{-1}S^{-1}\cdot \\ &\qquad \left(SD(\tilde{\alpha}_i)^{-1}S^{-1}+o(e^{s^{\frac{1}{3}}\tilde{\mu}^i})\right) \\ &-\prod_{i=1}^{\ell}SD(\delta_i)^{-1}U(\theta_i,\theta_{i+1})^{-1}D(\alpha_i)^{-1}S^{-1}.\end{aligned}$$

Hence $E(s)$ is a sum of terms in which at least one of the factors contains $o(\mathrm{Id})$, $o(e^{s^{\frac{1}{3}}\tilde{\mu}^i})$, or $o(e^{s^{\frac{1}{3}}\tilde{\lambda}^i})$. For instance, in the case where $o(\mathrm{Id})$ arises once, we will have a term of the form

$$\begin{aligned}E_m(s)=&\prod_{i=1}^{m-1}SD(\delta_i)^{-1}U(\theta_i,\theta_{i+1})^{-1}D(\alpha_i)^{-1}S^{-1}\cdot \\ &SD(\delta_m)^{-1}S^{-1}o(\mathrm{Id})SD(\alpha_m)^{-1}S^{-1}\cdot \\ &\prod_{i=m+1}^{\ell}SD(\delta_i)^{-1}U(\theta_i,\theta_{i+1})^{-1}D(\alpha_i)^{-1}S^{-1}\end{aligned}$$

and so

$$\|E_m(s)\|_\infty\le C\,o(1)\prod_{i=1}^{\ell}\|D(\delta_i)^{-1}\|_\infty\|D(\alpha_i)^{-1}\|_\infty\le C\,o(1)\prod_{i=1}^{\ell}e^{s^{\frac{1}{3}}\nu^i}=o(\|A(s)\|_\infty).$$

In this case, of course, we combined diagonal matrices in a convenient way. Yet even when the diagonal matrices do not simplify, we may obtain the same estimate. For instance, consider

$$\begin{aligned} E'_m(s) = \prod_{i=1}^{m-1} & SD(\delta_i)^{-1}U(\theta_i,\theta_{i+1})^{-1}D(\alpha_i)^{-1}S^{-1}\cdot \\ & o(e^{s^{\frac13}\tilde\lambda^m})SD(\delta_m\setminus\tilde\delta_m)^{-1}U(\theta_m,\theta_{m+1})^{-1}D(\alpha_m)^{-1}S^{-1} \\ \prod_{i=m+1}^{\ell} & SD(\delta_i)^{-1}U(\theta_i,\theta_{i+1})^{-1}D(\alpha_i)^{-1}S^{-1}. \end{aligned}$$

Then

$$\begin{aligned} \|E'_m(s)\|_\infty &\le C\,o(e^{s^{\frac13}\tilde\lambda^m})\|D(\delta_m\setminus\tilde\delta_m)^{-1}\|_\infty\|D(\alpha_m)^{-1}\|_\infty\cdot \\ &\quad \prod_{\substack{i=1\\ i\neq m}}^{\ell}\|D(\delta_i)^{-1}\|_\infty\|D(\alpha_i)^{-1}\|_\infty \\ &= C\,o(e^{s^{\frac13}\tilde\lambda^m})\prod_{\substack{i=1\\ i\neq m}}^{\ell} e^{s^{\frac13}(\mu^i+\lambda^i)}\|D(\delta_m\setminus\tilde\delta_m)^{-1}\|_\infty\|D(\alpha_m)^{-1}\|_\infty. \end{aligned}$$

Here we used Lemma 5.11 and the fact that $e^{s^{\frac13}\lambda^m}$ is the largest eigenvalue of $D(\delta_m)^{-1}$. This last expression will be $o(\|A(s)\|_\infty)$ if we can show that $o(e^{s^{\frac13}\tilde\lambda^m})\|D(\delta_m\setminus\tilde\delta_m)^{-1}\|_\infty = o(e^{s^{\frac13}\lambda^m})$. Note that $\|D(\alpha_m)^{-1}\|_\infty = e^{s^{\frac13}\mu_m}$. Now, if $\tilde\delta_m(t) = s^{\frac13}te^{i\theta_m}$ with $t\in[\epsilon, L_m/2]$, then $\delta_m(t) = s^{\frac13}te^{i\theta_m}$ with $t\in[0,L_m/2]$ and

$$\begin{aligned} D(\tilde\delta_m)^{-1} &= \exp\left(s^{\frac13}(\epsilon-L_m/2)\begin{pmatrix}\cos(\theta_m) & & \\ & \cos(\theta_m-\frac{2\pi}{3}) & \\ & & \cos(\theta_m-\frac{4\pi}{3})\end{pmatrix}\right), \\ D(\delta_m)^{-1} &= \exp\left(-s^{\frac13}L_m/2\begin{pmatrix}\cos(\theta_m) & & \\ & \cos(\theta_m-\frac{2\pi}{3}) & \\ & & \cos(\theta_m-\frac{4\pi}{3})\end{pmatrix}\right), \\ D(\delta_m\setminus\tilde\delta_m)^{-1} &= \exp\left(-s^{\frac13}\epsilon\begin{pmatrix}\cos(\theta_m) & & \\ & \cos(\theta_m-\frac{2\pi}{3}) & \\ & & \cos(\theta_m-\frac{4\pi}{3})\end{pmatrix}\right). \end{aligned}$$

So if the largest eigenvalue of $D(\tilde\delta_m)^{-1}$ is in position j, then

$$\begin{aligned} \|D(\tilde\delta_m)^{-1}\|_\infty &= e^{s^{\frac13}(\epsilon-L_m/2)\cos(\theta_m-(j-1)2\pi/3)} = e^{s^{\frac13}\tilde\lambda^m}, \\ \|D(\delta_m\setminus\tilde\delta_m)^{-1}\|_\infty &= e^{-s^{\frac13}\epsilon\cos(\theta_m-(j-1)2\pi/3)}, = e^{s^{\frac13}(\lambda^m-\tilde\lambda^m)} \\ \|D(\delta_m)^{-1}\|_\infty &= e^{-s^{\frac13}(L_m/2)\cos(\theta_m-(j-1)2\pi/3)} = e^{s^{\frac13}\lambda^m}, \end{aligned}$$

Thus, $o(e^{s^{\frac13}\tilde\lambda^m})\|D(\delta_m\setminus\tilde\delta_m)^{-1}\|_\infty = o(e^{s^{\frac13}\lambda^m})$, as required.

The remaining cases are analogous, involving smaller error terms. □

In particular, if we choose the submultiplicative matrix norm

$$\|M\|_2 := \sigma_1(M)$$

where $\sigma_1(M)$ denotes the highest singular value of M, in other words the highest eigenvalue of $\sqrt{M^t M}$, we obtain

Corollary 7.2. *For every regular path c_γ that is the concatenation of saddle connections $c_\ell, \dots, c_1$ we have*

$$\lim_{s\to+\infty} \frac{\log(\sigma_1(\mathrm{Hol}_s(c_\gamma)))}{s^{\frac{1}{3}}} = \sum_{i=1}^{l} \lim_{s\to+\infty} \frac{\log(\sigma_1(\mathrm{Hol}_s(c_i)))}{s^{\frac{1}{3}}} \ .$$

Proof. Because the paths c_γ and $\tilde{c}_\gamma$ have the same holonomy, by definition of $\mu^i_{k_i}$ and $\lambda^i_{j_i}$, Proposition 5.6 and the fact that the saddle connection c_i is the concatenation of δ_i and α_{i-1} (see Figure 1), we have

$$\lim_{s\to+\infty} \frac{\log(\sigma_1(\mathrm{Hol}_s(c_i)))}{s^{\frac{1}{3}}} = \mu^i_{k_i} + \lambda^{i-1}_{j_{i-1}} = \nu^i,$$

where the indices are to be intended modulo ℓ. Thus the result follows from Theorem 7.1 applied to the norm $\|\cdot\|_2$. □

We can also deduce the asymptotics of the other singular values

Corollary 7.3. *Let σ_j denote the j-th largest singular value. Then*

$$\lim_{s\to+\infty} \frac{\log(\sigma_j(\mathrm{Hol}_s(c_\gamma)))}{s^{\frac{1}{3}}} = \sum_{i=1}^{\ell} \lim_{s\to+\infty} \frac{\log(\sigma_j(\mathrm{Hol}_s(c_i)))}{s^{\frac{1}{3}}}$$

for $j = 1, 2, 3$.

Proof. We already know that the result holds for $\sigma_1(\mathrm{Hol}_s(c_\gamma))$ by Corollary 7.2. Because

$$\log(\sigma_3(\mathrm{Hol}_s(c_\gamma))) = -\log(\sigma_1(\mathrm{Hol}_s^{-1}(c_\gamma))) = \log(\sigma_1(\mathrm{Hol}_s(c_\gamma^{-1}))) \ ,$$

the statement is also true for $\sigma_3(\mathrm{Hol}_s(c_\gamma))$ by applying the previous corollary to the path c_γ^{-1}. Moreover, since $\mathrm{Hol}_s(c_\gamma) \in \mathrm{SL}(3, \mathbb{R})$, we have

$$\log(\sigma_1(\mathrm{Hol}_s(c_\gamma))) + \log(\sigma_2(\mathrm{Hol}_s(c_\gamma))) + \log(\sigma_3(\mathrm{Hol}_s(c_\gamma))) = 0 \ ,$$

hence the result holds for $\sigma_2(\mathrm{Hol}_s(c_\gamma))$ as well. □

We now give an argument extending the above results to the case of flat geodesics that are not regular, in that they contain segments in the wall directions of a Weyl chamber. We begin with the generic situation where each flat geodesic segment has corresponding diagonal holonomy with a unique largest eigenvalue. In that case, as above, we compute the largest asymptotic singular value and then reverse the direction of the path to find the smallest. This determines the asymptotic eigenvalue structure. In particular, the arguments above already suffice to determine the largest asymptotic singular value along any geodesic path in which each segment has a unique largest eigenvalue. We summarize the discussion in the following proposition.

Proposition 7.4. *Theorem 7.1 and Corollary 7.2 hold for flat geodesic paths all of whose segments c_i are such that each diagonal matrix $D(c_i)^{-1}$ has a distinct largest eigenvalue.*

Now we address the remaining case where paths may contain segments so that at least one $D(c_i)^{-1}$ has two largest eigenvalues. In this case, Lemma 5.11 becomes

$$A(s) = \prod_{i=1}^{\ell} e^{s^{\frac{1}{3}}(\mu^i_{j_i}+\lambda^i_{k_i})} S\Big(\sum_{\alpha} c_{j_{i_\alpha},k_{i_\alpha}} E_{j_{i_\alpha},k_{i_\alpha}}\Big)(\mathrm{Id}+o(\mathrm{Id}))S^{-1}, \tag{7.1}$$

where in each sum α ranges over one or two indices in $\{1,2,3\}$: one if there is a unique largest eigenvalue of $D(c_i)^{-1}$, and two if there are two largest eigenvalues. Proposition 6.8 then shows that the coefficients $c_{j_{i_\alpha},k_{i_\alpha}}$ are all positive, and thus there are no cancellations. This is enough for the analysis above on submultiplicative matrix norms and singular values to apply.

Thus we have proved

Theorem 7.5. *For every geodesic path $\tilde{c}_\gamma$, we have*

$$\lim_{s\to+\infty} \frac{\log \|\mathrm{Hol}_s(\tilde{c}_\gamma)\|}{s^{\frac{1}{3}}} = \sum_{i=1}^{\ell} \nu^i$$

where $\|\cdot\|$ is any submultiplicative matrix norm.
In particular, if $\sigma_j(\mathrm{Hol}_s(c_\gamma))$ denotes the j-th largest singular value of the flat geodesic c_γ homotopic to $\tilde{c}_\gamma$ with saddle connections $c_1,\dots,c_l$, then

$$\lim_{s\to+\infty} \frac{\log(\sigma_j(\mathrm{Hol}_s(c_\gamma)))}{s^{\frac{1}{3}}} = \sum_{i=1}^{\ell} \lim_{s\to+\infty} \frac{\log(\sigma_j(\mathrm{Hol}_s(c_i)))}{s^{\frac{1}{3}}}$$

for $j=1,2,3$.

Because $\mathrm{Hol}_s(c_\gamma)$ is diagonalizable with positive eigenvalues, Theorem 7.5 also implies a similar asymptotic formula for the eigenvalues of $\mathrm{Hol}_s(c_\gamma)$.

Corollary 7.6. *For every closed curve $\gamma\in\pi_1(S)$, let c_γ be the geodesic representative for the flat metric $|q_0|^{\frac{2}{3}}$. Assume that c_γ is the concatenation of saddle connections $c_1,\dots,c_l$. Then*

$$\lim_{s\to+\infty} \frac{\log(\Lambda(\mathrm{Hol}_s(\gamma)))}{s^{\frac{1}{3}}} = \sum_{i=1}^{\ell} \lim_{s\to+\infty} \frac{\log(\Lambda(\mathrm{Hol}_s(c_i)))}{s^{\frac{1}{3}}} = \sum_{i=1}^{\ell} \nu^i$$

where $\Lambda(M)$ denotes the spectral radius of M.
In particular, Corollary 7.3 holds when replacing singular values with eigenvalues.

Proof. It is well known that the spectral radius, i.e. the absolute values of the largest (possibly complex) eigenvalue of a matrix M, can be computed as

$$\Lambda(M) = \lim_{r \to +\infty} \|M^r\|^{\frac{1}{r}} .$$

Since $\mathrm{Hol}_s(\tilde{c}_\gamma)$ has all real and positive eigenvalues, its spectral radius coincides with its largest eigenvalue. Moreover, because c_γ and $\tilde{c}_\gamma$ are in the same free homotopy class, we can do this computation for $\mathrm{Hol}_s(\tilde{c}_\gamma)$. Now, we know that

$$\mathrm{Hol}_s(\tilde{c}_\gamma) = A(s) + E(s)$$

with $\|E(s)\| = o(\|A(s)\|)$ as $s \to +\infty$ and

$$A(s) = \prod_{i=1}^{\ell} e^{s^{\frac{1}{3}}(\mu^i_{j_i} + \lambda^i_{k_i})} S\Big(\sum_\alpha c_{j_{i_\alpha}, k_{i_\alpha}} E_{j_{i_\alpha}, k_{i_\alpha}}\Big)(\mathrm{Id} + o(\mathrm{Id}))S^{-1} \tag{7.2}$$

for some positive constants $c_{j_{i_\alpha}, k_{i_\alpha}}$. Fix $\delta > 0$ small and let s_0 be such that for all $s \geq s_0$ we have $\|E(s)\| \leq \delta \|A(s)\|$. Then, for all $s \geq s_0$

$$\mathrm{Hol}_s(\tilde{c}_\gamma)^r = A(s)^r + E'(s)$$

with $\|E'(s)\| \leq 2r\delta\|A(s)\|^r$ for every integer $r > 1$. Therefore,

$$\begin{aligned} \Lambda(\mathrm{Hol}_s(\tilde{c}_\gamma)) &= \lim_{r \to +\infty} \|\mathrm{Hol}_s(\tilde{c}_\gamma)^r\|^{\frac{1}{r}} \\ &= \lim_{r \to +\infty} \|A(s)^r\|^{\frac{1}{r}} \left\| \frac{A(s)^r}{\|A(s)^r\|} + \frac{E'(s)}{\|A(s)^r\|} \right\|^{\frac{1}{r}} \\ &= \Lambda(A(s))C(s) \end{aligned}$$

for all $s \geq s_0$, where

$$C(s) = \lim_{r \to +\infty} \left\| \frac{A(s)^r}{\|A(s)^r\|} + \frac{E'(s)}{\|A(s)^r\|} \right\|^{\frac{1}{r}} .$$

First we compute the spectral radius $\Lambda(A(s))$ of the matrix $A(s)$. From (7.2), we find

$$\begin{aligned} A(s)^r &= \exp\left(\sum_{i=1}^{\ell} r s^{\frac{1}{3}} \nu^i\right) \left(\prod_{i=1}^{\ell} S\left(\sum_\alpha c_{j_{i_\alpha}, k_{i_\alpha}} E_{j_{i_\alpha}, k_{i_\alpha}}\right)(\mathrm{Id} + o(\mathrm{Id}))S^{-1}\right)^r \\ &= \exp\left(\sum_{i=1}^{\ell} r s^{\frac{1}{3}} \nu^i\right) (M + o(\mathrm{Id}))^r \end{aligned}$$

as $s \to +\infty$, where

$$M = \prod_{i=1}^{\ell} S\left(\sum_\alpha c_{j_{i_\alpha}, k_{i_\alpha}} E_{j_{i_\alpha}, k_{i_\alpha}}\right) S^{-1} \neq 0$$

Hence, for s sufficiently large,

$$\Lambda(A(s)) = \lim_{r\to+\infty} \|A(s)^r\|^{\frac{1}{r}} = \exp\left(\sum_{i=1}^{\ell} s^{\frac{1}{3}}\nu^i\right)\Lambda(M + o(\mathrm{Id})) \ .$$

We then observe that the function $C(s)$ is uniformly bounded for $s \geq s_0$, because

$$\begin{aligned}\left\|\frac{A(s)^r}{\|A(s)^r\|} + \frac{E'(s)}{\|A(s)^r\|}\right\| &\leq 1 + \frac{\|E'(s)\|}{\|A(s)^r\|} \\ &\leq 1 + \frac{2r\delta\|A(s)\|^r}{\|A(s)^r\|} \\ &\leq \frac{(1+2r\delta)\|M+o(\mathrm{Id})\|^r}{\|(M+o(\mathrm{Id}))^r\|}\end{aligned}$$

which implies that $C(s) \leq \|M + o(\mathrm{Id})\|\Lambda(M + o(\mathrm{Id}))^{-1}$. Therefore,

$$\lim_{s\to+\infty} \frac{\log(\Lambda(\mathrm{Hol}_s(\tilde{c}_\gamma)))}{s^{\frac{1}{3}}} = \lim_{s\to+\infty} \frac{\log(\Lambda(A(s)))}{s^{\frac{1}{3}}} = \sum_{i=1}^{\ell} \nu^i \ .$$

Now, the saddle connection c_i is the concatenation of δ_i and α_{i-1}, where indices are intended modulo ℓ (see Figure 1), so by Theorem 4.6,

$$\lim_{s\to+\infty} \frac{\log(\Lambda(\mathrm{Hol}_s(c_i)))}{s^{\frac{1}{3}}} = \nu^i \ ,$$

which gives the desired asymptotics of the largest eigenvalue. Repeating the same argument as in Corollary 7.3, the formula actually holds for all eigenvalues of $\mathrm{Hol}_s(\gamma)$. □

8. Harmonic map to the real building

By work of Hitchin ([Hit92]), the Hitchin representations $\rho_s : \pi_1(S) \to \mathrm{SL}(3,\mathbb{R})$ arising from the ray of cubic differentials $q_s = sq_0$ are constructed along with an associated ρ_s-equivariant conformal harmonic map $h_s : \tilde{\Sigma} \to \mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$ to the symmetric space (see Section 2). In this section we study both the asymptotic behavior of h_s around a zero and also describe the geometry of the limiting harmonic map $h_\infty : \tilde{\Sigma} \to \mathcal{B}$ to an $\mathbb{R}$-building.

8.1. **Generalities on Euclidean buildings.** We recall here the definition and main properties of $\mathbb{R}$-buildings. We direct the interested reader to [KL97] for a more thorough discussion.

Let $\mathbb{A}$ denote a finite-dimensional affine Euclidean space. The Tits boundary of $\mathbb{A}$ is a sphere, denoted by $\partial_{Tits}\mathbb{A}$. A subgroup $W_{aff} \subset \mathrm{Isom}(\mathbb{A})$ is an affine Weyl group if it is generated by reflections across hyperplanes of $\mathbb{A}$, called walls, and its linear part W_{lin} is finite. The pair $(\mathbb{A}, W_{aff})$ is a Euclidean Coxeter complex. We denote by Δ_{mod} the quotient $\partial_{Tits}\mathbb{A}/W_{lin}$.

An oriented geodesic ray determines a point in $\partial_{Tits}\mathbb{A}$. Its Δ_{mod}-direction is its projection to Δ_{mod}. A *Weyl chamber* with tip at $p \in \mathbb{A}$ is a complete cone with vertex at p for which its Tits boundary is a Δ_{mod} chamber. A germ of a Weyl chamber based at $p \in \mathbb{A}$ is an equivalence class of Weyl chambers based at p for the following equivalence relation: W and W' are equivalent if their intersection is a neighborhood of p in both W and W'. The germ of a Weyl chamber is denoted by $\Delta_p W$. We say that two germs $\Delta_p W$ and $\Delta_p W'$ are opposite if one is the image of the other under the longest element in the Weyl group W_{lin}. We say that two Weyl chambers based at p are opposite if their germs are.

Definition 8.1. A Euclidean $\mathbb{R}$-building modeled on a Euclidean Coxeter complex $(\mathbb{A}, W_{aff})$ is a $CAT(0)$ space $\mathcal{B}$ that satisfies the following axioms:

a) Each oriented geodesic segment $\overline{xy}$ is assigned a Δ_{mod}-direction $\theta(\overline{xy}) \in \Delta_{mod}$. For any pair of oriented geodesic segments $\overline{xy}$ and $\overline{xz}$ emanating from the same point $x \in \mathcal{B}$, the difference of their Δ_{mod}-directions is smaller than their comparison angle;
b) Given $\delta_1, \delta_2 \in \Delta_{mod}$, denote by $D(\delta_1, \delta_2)$ the finite set given by all of the possible distances between points in their W_{aff} orbit. The angle between any two geodesic segments $\overline{xy}$ and $\overline{xz}$ lies in the finite set $D(\theta(\overline{xy}), \theta(\overline{xz}))$.
c) There is a collection $\mathcal{A}$ of isometric embeddings $\iota_A : \mathbb{A} \to \mathcal{B}$ that preserve Δ_{mod}-directions and that is closed under precomposition by isometries in W_{aff}. Each image $A = \iota_A(\mathbb{A})$ is called an apartment of $\mathcal{B}$. Each geodesic segment, ray and complete geodesic is contained in an apartment.
d) Coordinate charts $\{\iota_A\}_{A \in \mathcal{A}}$ are compatible in the sense that, when defined, $\iota_{A_1} \circ \iota_{A_2}^{-1}$ is the restriction of an isometry in the Weyl group W_{aff}.

Remark 8.2. Many different, though equivalent, sets of axioms of Euclidean $\mathbb{R}$-buildings appear in the literature. For a detailed discussion we refer the reader to [BS14].

It follows immediately from the axioms that any two points $x, y \in \mathcal{B}$ are contained in a common apartment A and the distance between them coincides with the Euclidean distance computed inside A. Moreover, we can define a (germ of a) Weyl chamber in $\mathcal{B}$ as the image of a (germ of a) Weyl chamber in $\mathbb{A}$ under some chart ι_A. The following property will be useful.

Proposition 8.3 ([Par12])**.** *Two opposite Weyl chambers based at $p \in \mathcal{B}$ are contained in a unique apartment.*

The boundary at infinity of $\mathcal{B}$ is defined as the set of equivalence classes of geodesic rays, where two rays are equivalent if they remain at bounded distance. Given any $\xi \in \partial_\infty \mathcal{B}$ and $p \in \mathcal{B}$, there is a unique geodesic ray ξ_p in the equivalence class of ξ starting at p.

Given a point $p \in \mathcal{B}$, and two geodesic segments $c_1, c_2; [0,1] \to \mathcal{B}$ such that $c_j(0) = p$, the angle between them is the quantity

$$\angle_p(c_1, c_2) = \lim_{s,t \to 0} \widetilde{\angle}_p(c_1(s), c_2(t))$$

where $\widetilde{\angle}_p$ denotes the angle of the Euclidean comparison triangle. This induces a distance on the set $\Sigma_p\mathcal{B}$ of equivalence classes of geodesic segments emanating from p, where two segments are identified if the angle between them is zero.

8.2. **Asymptotic cone of** $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$**.** We denote by (X, d) the symmetric space $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$ endowed with the distance induced by its homogeneous Riemannian metric. The construction of the asymptotic cone of (X, d) and, more generally, of any metric space relies on the choice of a non-principal ultrafilter.

Definition 8.4. A non-principal ultrafilter is a finitely additive probability measure ω on $\mathcal{P}(\mathbb{R})$ such that

(1) $\omega(S) \in \{0, 1\}$ for every $S \in \mathcal{P}(\mathbb{R})$;
(2) $\omega(S) = 0$ for every finite subset S.

As we are interested in the behavior as $s \to +\infty$, we only consider non-principal ultrafilters each supported on a countable subset of $\mathbb{R}$ whose only limit point in $[-\infty, +\infty]$ is $+\infty$. Non-principal ultafilters allow us to consistently define limits of bounded sequences without passing to subsequences. Precisely, a family of points $\{y_s\}_{s\in\mathbb{R}}$ in a topological space Y is said to have a ω-limit y, denoted by $y = \lim_\omega y_s$ if for each neighborhood $\mathcal{U}$ of y we have $\omega(\{s \in \mathbb{R} \mid y_s \in \mathcal{U}\}) = 1$.

Definition 8.5. Let $*$ be a base point in (X, d) and let $\lambda_s \to \infty$ be a sequence of scaling factors. Fix a non-principal ultrafilter ω. The asymptotic cone of $(X, \lambda_s^{-1}d, *)$ is the metric space $(\mathrm{Cone}_\omega(X, \lambda_s, *), d_\omega)$ where

(1) points in $\mathrm{Cone}_\omega(X, \lambda_s, *)$ are equivalence classes of families $x_s \in X$ such that $\lambda_s^{-1}d(x_s, *)$ is bounded. Here, two families x_s, y_s are equivalent if $\lim_\omega \lambda_s^{-1}d(x_s, y_s) = 0$;
(2) the distance between two points $[x_s]$ and $[y_s]$ is defined as

$$d_\omega([x_s], [y_s]) := \lim_\omega \lambda_s^{-1}d(x_s, y_s) \ .$$

By work of [Tho02], the asymptotic cone of the symmetric space (X, d) is actually, up to isometries, independent of the choice of the ultrafilter ω (if we assume the continuum hypothesis) and of the base point $*$. Moreover, the asymptotic cone of the symmetric space (X, d) can also be interpreted as the Gromov-Hausdorff limit of the pointed sequence of metric spaces $X_s = (X, \lambda_s^{-1}d, *)$ ([Gro81] [KL97]) and it is a non-discrete Euclidean building modelled on the affine Weyl group of $\mathfrak{sl}(3,\mathbb{R})$. In particular, we are going to identify the model Euclidean plane $\mathbb{A}$ with

$$\mathbb{A} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\} \ .$$

Then the linear part of the Weyl group consists of reflections across the walls of equation $x_i - x_j = 0$ for all $i \neq j$. In particular, we identify Δ_{mod} with the boundary at infinity of the Weyl chamber

$$\mathfrak{a}^+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 > x_2 > x_3\} .$$

Given $x, y \in \mathrm{Cone}_\omega(X)$, we can find an apartment $\iota_A : \mathbb{A} \to \mathrm{Cone}_\omega(X)$ such that $\iota_A(0) = x$ and $\iota_A^{-1}(y) \in \mathfrak{a}^+$. Then the distance between x and y is

$$d_\omega(x, y) = d_{\mathbb{A}}(0, \iota_A^{-1}(y)) = \sqrt{y_1^2 + y_2^2 + y_3^2} ,$$

where y_i are the coordinates of $\iota_A^{-1}(y)$.

Beside the Euclidean distance on an apartment, it is also useful to consider the $\mathfrak{a}^+$-valued distance defined by

$$d_\omega^{\mathfrak{a}^+}(x, y) = d_{\mathbb{A}}^{\mathfrak{a}^+}(0, \iota_A^{-1}(y)) = (y_1, y_2, y_3) .$$

By a theorem of Parreau ([Par12]), the $\mathfrak{a}^+$-valued distance on $\mathrm{Cone}_\omega(X)$ is the ω-limit of the analogously defined $\mathfrak{a}^+$-valued distance $d^{\mathfrak{a}^+}$ on X rescaled by λ_s^{-1}. (Here this distance $d^{\mathfrak{a}^+}(x, y)$ on X relies on finding a flat that contains x and y.) In other words, if $x = [x_s], y = [y_s] \in \mathrm{Cone}_\omega(X)$ with $x_s, y_s \in X$, then

$$d_\omega^{\mathfrak{a}^+}(x, y) = \lim_\omega \lambda_s^{-1} d^{\mathfrak{a}^+}(x_s, y_s) .$$

An apartment in $\mathrm{Cone}_\omega(X)$ can be obtained as the ω-limit of a sequence of flats in X. Precisely, if $\iota_{F_s} : \mathbb{A} \to F_s \subset X$ are isometric parametrizations of a sequence of maximal flats F_s of X with the property that $d(F_s, *) = O(\lambda_s)$, then the family ι_{F_s} has an ω-limit $\iota_F : \mathbb{A} \to \mathrm{Cone}_\omega(X)$ which defines an apartment in $\mathrm{Cone}_\omega(X)$. See [KL97] for more details.

A family $G_s \in \mathrm{SL}(3, \mathbb{R})$ of isometries of X also induces an isometric action on $\mathrm{Cone}_\omega(X)$ provided that $d(G_s(*), *) = O(\lambda_s)$ by setting

$$G_s \cdot [x_s] := [G_s(x_s)]$$

for any $[x_s] \in \mathrm{Cone}_\omega(X)$.

8.3. **Limiting harmonic map to the building.** Given a ray of cubic differentials sq_0, we consider the family of conformal harmonic maps $h_s : \tilde{\Sigma} \to X$ that are equivariant under the corresponding Hitchin representations $\rho_s : \pi_1(\Sigma) \to \mathrm{SL}(3, \mathbb{R})$. We fix a non-principal ultrafilter ω, a base point $* \in X$ and the sequence of scaling factors $\lambda_s = s^{\frac{1}{3}}$. We can consider the maps h_s to take values in the re-scaled pointed metric spaces $X_s = ((X, *), s^{-\frac{1}{3}} d)$. The following result is due to Semin Kim (see also [SS25, Proposition 8.8]), but we include a sketch of the proof to justify our choice of scaling factors.

Proposition 8.6 ([Kim17],[SS25]). *The family $h_s : \tilde{\Sigma} \to X_s$ converges to a Lipschitz equivariant harmonic map $h_\infty : \tilde{\Sigma} \to \mathrm{Cone}_\omega(X)$. The family of holonomy maps $\rho_s : X_s \to X_s$ ω-converges to an isometry ρ_∞ of $\mathrm{Cone}_\omega(X)$, and h_∞ is equivariant with respect to ρ_∞.*

Sketch of proof. By [DM06, Theorem 1.2] and [Kim17, Theorem 3.1.2], it is sufficient to show that energy of the maps $h_s : \tilde{\Sigma} \to X$ grows as $O(s^{\frac{2}{3}})$. Since h_s is conformal, this amounts to estimating the area of $h_s(D)$, where $D \subset \tilde{\Sigma}$ is a compact fundamental domain for the action of $\pi_1(S)$. Now, the induced metric $\hat{g}_s$ on the minimal surfaces $h_s(\tilde{\Sigma})$ can be written in terms of the Blaschke metric g_s as (see [DL19])

$$\hat{g}_s = 2\left(1 + \frac{e^{-3\mathcal{F}_s}}{2}\right) g_s \ , \tag{8.1}$$

hence $\hat{g}_s$ is uniformly bi-Lipshitz to g_s and the result follows from Lemma 3.2.

The ω-convergence of the holonomy maps follows from Remark 3.19 in [Par12]. The ρ_∞-equivariance of h_∞ follows from the fact that each h_s is equivariant with respect to ρ_s. □

The behavior of the limiting harmonic map $h_\infty : \tilde{\Sigma} \to \mathrm{Cone}_\omega(X)$ is well-known outside the zeros of the cubic differential q_0 ([Moc16, Theorem 2.17]).

Theorem 8.7. *For any $p \in \tilde{\Sigma}$ that is not a lift of a zero of the cubic differential q_0, there is a neighborhood $\mathcal{U}_p$ centered at p and an apartment $\iota_A : \mathbb{A} \to \mathrm{Cone}_\omega(X)$ with $\iota_A(0) = h_\infty(p)$ such that*

i) the induced distance on $h_\infty(\mathcal{U}_p)$ is $\sqrt{3} \cdot 2^{\frac{1}{6}} d_{q_0}$;
ii) the limiting harmonic map h_∞ sends $\mathcal{U}_p$ inside $A = \iota_A(\mathbb{A})$;
iii) for any $q \in \mathcal{U}_p$ we have

$$\iota_A^{-1} \circ h_\infty(q) = \left(-2^{\frac{2}{3}}\mathcal{R}e\left(\int_p^q \phi_1\right), -2^{\frac{2}{3}}\mathcal{R}e\left(\int_p^q \phi_2\right), -2^{\frac{2}{3}}\mathcal{R}e\left(\int_p^q \phi_3\right)\right)$$

where ϕ_i are the cube roots of q_0 (which are well-defined in $\mathcal{U}_p$).

Proof. Let $\mathcal{U}_p$ be a q_0-disk around p that avoids neighborhoods of zeroes of q_0. From Equation 8.1 and Lemma 3.4, we know that the induced metric $\hat{g}_s$ rescaled by $s^{-\frac{2}{3}}$ converges to $3 \cdot 2^{\frac{1}{3}}|\tilde{q}_0|^{\frac{2}{3}}$ uniformly on $h_\infty(\mathcal{U}_p)$. To conclude that h_∞ sends $\mathcal{U}_p$ inside a single apartment, it is sufficient to show that $h_\infty(\mathcal{U}_p)$ is totally geodesic: this follows from an extendibility feature of flat neighborhoods in buildings (see [AB08, Theorem 11.53]). To this aim we show that for every $q, q' \in \mathcal{U}_p$ we have that

$$d_\omega(h_\infty(q), h_\infty(q')) = \sqrt{3} \cdot 2^{\frac{1}{6}} d_{q_0}(q, q'). \tag{8.2}$$

Since the asymptotic cone is a $CAT(0)$ space, this implies that the unique geodesic connecting $h_\infty(q)$ and $h_\infty(q')$ is entirely contained in the image of h_∞, hence $h_\infty(\mathcal{U}_p)$ is totally geodesic inside $\mathrm{Cone}_\omega(X)$. We are thus left to prove Equation 8.2. Fix a natural coordinate w on $\mathcal{U}_p$ and let w_0 and w_0' be the coordinates of q and q' in this chart. We then parametrize the geodesic connecting w_0 and w_0' as $\gamma(t) = w_0 + te^{i\theta}$ with $t \in [0, L]$ so that $w_0' = w_0 + Le^{i\theta}$. Recall that the map h_∞ is the ω-limit of the maps $h_s : \tilde{\Sigma} \to X_s$ that can be expressed as

$$h_s(w) = P_s(0)F_s(w)P_s(w)^{-1}$$

for some $P_s \in \mathrm{SL}(3,\mathbb{C})$. Indeed, from Section 2, we know that the equivariant harmonic map h_s is simply given by the frame field F_s of the affine sphere $f_s : \mathbb{C} \to \mathbb{R}^3$ whose columns form at each point $w \in \mathbb{C}$ a real basis of $\mathbb{R}^3$ that is orthonormal for (the lift of) the Blaschke metric. The matrices P_s represent the change of frame between a real othonormal basis and the basis $\{1, \partial_w, \partial_{\bar{w}}\}$ induced by the natural coordinate w. Therefore,

$$\begin{aligned} d_\omega(h_\infty(w_0), h_\infty(w_0')) &= \lim_{s\to+\infty} s^{-\frac{1}{3}} d(h_s(w_0), h_s(w_0)') \\ &= \lim_{s\to+\infty} s^{-\frac{1}{3}} d(F_s(w_0)P_s^{-1}(w_0), F_s(w_0')P_s(w_0')^{-1}) \\ &= \lim_{s\to+\infty} s^{-\frac{1}{3}} d(P_s(w_0')F_s(w_0')^{-1}F_s(w_0)P_s^{-1}(w_0), \mathrm{Id}) \\ &= \lim_{s\to+\infty} s^{-\frac{1}{3}} d(\mathrm{Hol}_s(\gamma), \mathrm{Id}) \end{aligned} \tag{8.3}$$

By Proposition 5.6 (see also [Lof07]) and Corollary 7.3 the singular values $\sigma_j(s)$ of $\mathrm{Hol}_s(\gamma)$ satisfy

$$\lim_{s\to+\infty} s^{-\frac{1}{3}} \log(\sigma_j(s)) = -2^{\frac{2}{3}} L\lambda_j$$

where $(\lambda_1, \lambda_2, \lambda_3)$ is a reordering of $(\cos(\theta), \cos(\theta - 2\pi/3), \cos(\theta - 4\pi/3))$ such that $\lambda_1 \geq \lambda_2 \geq \lambda_3$. Therefore, using that the distance d in the symmetric space X from the identity is given as the Euclidean distance to the logarithms of the singular values, we see

$$\begin{aligned} d_\omega(h_\infty(w_0), h_\infty(w_0')) &= \lim_{s\to+\infty} s^{-\frac{1}{3}} \sqrt{\sum_{j=1}^{3} \log^2(\sigma_j(s))} \\ &= \lim_{s\to+\infty} s^{-\frac{1}{3}} \sqrt{\sum_{j=1}^{3} s^{\frac{2}{3}} 2^{\frac{4}{3}} L^2 \lambda_j^2 + o(s^{\frac{2}{3}})} \\ &= 2^{\frac{2}{3}} L \sqrt{\sum_{j=1}^{3} \cos^2\left(\theta - \frac{2\pi(j-1)}{3}\right)} \\ &= \sqrt{3} \cdot 2^{\frac{1}{6}} L\ . \end{aligned}$$

Since $L = d_{q_0}(q, q')$ the proof of Equation 8.2 is complete.
Part *iii*) is a direct consequence of part *ii*) and the fact that the coordinates inside an apartment are given by the rescaled limit of the singular values of $h_s(w)$. □

We note, in particular, that outside the zeros of q_0 the map h_∞ is smooth, so the set of its singular points is discrete, and is locally injective. The following result about how these flats combine outside the zeros can be found in [KNPS15, Proposition 3.18]; we include the proof for the convenience of the reader.

Proposition 8.8. *Let $\gamma : [0,1] \to \Sigma$ be a geodesic path which avoids all zeros of q_0 and which is not in the direction of a wall of the Weyl chamber. Then there is an $A \subset \mathrm{Cone}_\omega(X)$ such that $h_\infty(\gamma(t)) \in A$ for all $t \in [0,1]$.*

Proof. Let $J = \{t \in I \mid$ there is apartment $A \subset \mathrm{Cone}_\omega(X)$ such that $h_\infty(\gamma([0,t])) \in A\}$. We want to show that $J = I$. First, we note that J is not empty because there is an apartment containing $h_\infty(\gamma(0))$ by axiom c) in the definition of buildings. Moreover, it is clear that if $t \in J$ and $s \leq t$ then $s \in J$. Let $t_0 = \sup(J)$ and suppose by contradiction that $t_0 < 1$. Let $q = \gamma(t_0)$. By Theorem 8.7, there is a neighborhood $\mathcal{U}_q \subset \Sigma$ and an apartment A_q such that $h_\infty(\mathcal{U}_q) \subset A_q$. Up to choosing a smaller $\mathcal{U}_q$, we can assume that $\mathcal{U}_q \cap \gamma(I) = \gamma(J_0)$ for some open interval J_0 containing t_0. Let $t_1 \in J_0 \cap J$ and let $p = \gamma(t_1)$. Because $t_1 < t_0$, there is an apartment $A_p \subset \mathrm{Cone}_\omega(X)$ such that $h_\infty(\gamma([0,t_1])) \subset A_p$. Let W_p^- denote the Weyl chamber with tip at $x = h_\infty(p)$ containing $h_\infty(\gamma([0,t_1]))$. Since $h_\infty(\gamma(J_0)) \subset A_q$, we can find two opposite Weyl chambers $W_q^\pm$ with tip at x such that $h_\infty(\gamma(t)) \in W_q^+$ for $t \geq t_1$ and $h_\infty(\gamma(t)) \in W_q^-$ for $t \leq t_1$. Note that the germs of the sectors W_p^- and W_q^+ are opposite because the Weyl chambers W_p^- and W_q^- are equivalent and W_q^+ is clearly opposite to W_q^-. Hence, by Proposition 8.3, there is a unique apartment $A \subset \mathcal{B}$ that contains W_p^- and W_q^+. Therefore, we can find $t_2 \in J_0 \cap J$ with $t_2 > t_0$ such that $h_\infty(\gamma([0,t_2])) \subset A$. Hence $t_2 \in J$ contradicting the fact that $t_0 = \sup(J)$. □

Remark 8.9. The restriction to geodesics is not important; all that is required is that the tangent vectors to the path all lie in the interior of a single Weyl chamber and the path avoids all zeros of q_0.

We intend to complete the description of h_∞ by studying its behavior in a neighborhood of a zero of the cubic differential. Let us fix a coordinate chart around a zero of order k of q_0 such that $q_0 = z^k dz^3$ on the ball $B = \{|z| < \epsilon\}$.

Theorem 8.10. *The image $h_\infty(B)$ consists of the union of $2(k+3)$ (cyclically ordered) bounded, closed sectors $\{W_i\}_{i=1}^{2(k+3)}$ of angle $\frac{\pi}{3}$ with tip at $x = h_\infty(0)$ such that*

i) $W_i \cap W_j = \emptyset$ for all $j \neq i \pm 1$ (with indices intended modulo $2(k+3)$);
ii) $W_i \cap W_{i+1}$ is a geodesic segment.

Proof. Let $\epsilon > 0$ small and denote by ξ_i for $i = 1, \dots, 2(k+3)$ the Stokes directions emanating from $0 \in B$. Let C_i be the sector in B with tip at 0 bounded by the directions $\xi_i + \epsilon$ and $\xi_{i+1} - \epsilon$. The ball B can be covered by $(k+3)$ standard half-planes obtained from the natural coordinates $w = \frac{3}{k+3} z^{\frac{k+3}{3}}$. Note that two such half-planes intersect in a sector of angle $\frac{\pi}{3}$ and in these coordinates the Stokes directions correspond to the angles $\pm\frac{\pi}{6}$ and $\pm\frac{\pi}{2}$. By Corollary 4.4, we can write

$$h_s(w) = P_s(0) A_i (\mathrm{Id} + o(\mathrm{Id})) F_T(s^{\frac{1}{3}} w) P_s(w)^{-1} \ . \tag{8.4}$$

Since

$$F_T(w) = S\ \exp\begin{pmatrix} 2^{\frac{2}{3}}\mathcal{R}e(w) & 0 & 0 \\ 0 & 2^{\frac{2}{3}}\mathcal{R}e(w/\omega) & 0 \\ 0 & 0 & 2^{\frac{2}{3}}\mathcal{R}e(w/\omega^2) \end{pmatrix} S^{-1}\ ,$$

by the same argument as in the proof of Theorem 8.7 the ω-limit h_T of the map $P_s(0)(\mathrm{Id}+o(\mathrm{Id}))F_T(s^{\frac{1}{3}}w)P_s(w)^{-1}$ sends C_i inside an apartment in $\mathrm{Cone}_\omega(X)$. Moreover, the image of a radial path in C_i is a geodesic in the building. Indeed, if we parameterize such a path by $\gamma(t) = te^{i\theta}$ with $t \in [0, L]$ in the natural coordinate w, then for all $t \in [0, L]$ we have as in (8.3)

$$\begin{aligned} d_\omega^{\mathfrak{a}^+}(h_T(0)), h_T(\gamma(t)) &= \lim_{s\to+\infty} s^{-\frac{1}{3}} d^{\mathfrak{a}^+}(\mathrm{Id}, P_s(0)(\mathrm{Id}+o(\mathrm{Id}))F_T(s^{\frac{1}{3}}te^{i\theta})P_s(w)^{-1}) \\ &= t\, 2^{\frac{2}{3}}(\lambda_1, \lambda_2, \lambda_3) \end{aligned} \tag{8.5}$$

where $(\lambda_1, \lambda_2, \lambda_3)$ is a permutation of $(\cos(\theta), \cos(\theta - 2\pi/3), \cos(\theta - 4\pi/3))$ such that $\lambda_1 \geq \lambda_2 \geq \lambda_3$. Because this reordering only depends on θ, which is constant along a radial path, and we already know that the image of h_T is entirely contained in a flat, we deduce that the image $h_T(\gamma)$ is a straight segment of length

$$\begin{aligned} d_\omega(h_T(0), h_T(\gamma(L))) &= 2^{\frac{2}{3}}L\sqrt{\sum_{j=1}^{3}\cos^2\left(\theta - \frac{2(j-1)\pi}{3}\right)} \\ &= \sqrt{3}\cdot 2^{\frac{1}{6}}L \end{aligned}$$

Since ϵ is arbitrary and h_∞ is continuous, we can conclude that the image of each open sector between two consecutive Stokes rays must be contained in a closed sector W_i with tip at $x = h_\infty(0)$ of angle $\frac{\pi}{3}$.
Now, recalling that $A_i^{-1}A_j = SU_{i,j}S^{-1}$ for some product of unipotents $U_{i,j}$, we can write

$$h_{\infty|_{C_j}} = P_s(0)A_iSU_{i,j}S^{-1}h_TP_s(0)^{-1}\ .$$

We first use this to show that the interiors of W_i and W_{i+1} are disjoint. This immediately implies that $W_i \cap W_{i+1}$ is a geodesic segment because h_∞ is continuous and each W_i is circular sector. The previous computation (8.5) about the behavior of radial paths under h_∞ shows that we may have $h_\infty(w_i) = h_\infty(w_{i+1})$ for some $w_i \in C_i$ and $w_{i+1} \in C_{i+1}$ only if they are at the same distance from the zero. However, in this present case we compute from the expression $A_i^{-1}A_j = SU_{i,j}S^{-1}$ and Equation 8.4,

$$
\begin{aligned}
& d_\omega(h_\infty(w_i), h_\infty(w_{i+1})) \\
= \; & \lim_{s\to+\infty} s^{-\frac{1}{3}} d(F_T(s^{\frac{1}{3}}w_i)P_s(w_i)^{-1}, (\mathrm{Id}+o(\mathrm{Id}))SU_{i,i+1}S^{-1}F_T(s^{\frac{1}{3}}w_{i+1})P_s(w_{i+1})^{-1}) \\
= \; & \lim_{s\to+\infty} s^{-\frac{1}{3}} d(P_s(w_{i+1})F_T^{-1}(s^{\frac{1}{3}}w_{i+1})SU_{i,i+1}^{-1}S^{-1}(\mathrm{Id}+o(\mathrm{Id}))F_T(s^{\frac{1}{3}}w_i)P_s(w_i)^{-1}, \mathrm{Id}) \\
\geq \; & \lim_{s\to+\infty} s^{-\frac{1}{3}} \log \left\| \frac{\sqrt{3}}{3} P_s(w_{i+1})F_T^{-1}(s^{\frac{1}{3}}w_{i+1})SU_{i,i+1}^{-1}S^{-1}(\mathrm{Id}+o(\mathrm{Id}))F_T(s^{\frac{1}{3}}w_i)P_s(w_i)^{-1} \right\|_\infty \\
= \; & \lim_{s\to+\infty} s^{-\frac{1}{3}} \log \| P_s(w_{i+1})SD^{-1}(s^{\frac{1}{3}}w_{i+1})U_{i,i+1}^{-1}(\mathrm{Id}+o(\mathrm{Id}))D(s^{\frac{1}{3}}w_i)S^{-1}P_s(w_i)^{-1}\|_\infty
\end{aligned}
$$

where we use again that the distance d in the symmetric space X from the identity is given as the Euclidean distance to the logarithms of the singular values. Now, note that in $D^{-1}(s^{\frac{1}{3}}w_{i+1})U_{i,i+1}^{-1}D(s^{\frac{1}{3}}w_i)$, at least one element on the diagonal is of the form $e^{cs^{\frac{1}{3}}}$ for some $c > 0$: here we use that we can express $D(s^{\frac{1}{3}}w_{i+1})$ and $D(s^{\frac{1}{3}}w_i)$ in a single coordinate, observing that the matrices are distinct, as well as that the unipotent has but a single off-diagonal nonzero entry. Hence,

$$d_\omega(h_\infty(w_i), h_\infty(w_{i+1})) \geq c > 0$$

and $h_\infty(w_i) \neq h_\infty(w_{i+1})$.
The same argument shows that $h_\infty(C_i)$ and $h_\infty(C_j)$ are disjoint as long as $i \neq j$ and the natural coordinates w_i and w_j do not coincide. Note that in this case the bound

$$d_\omega(h_\infty(w_i), h_\infty(w_{i+1})) \geq c$$

can be made independent of ϵ as the diagonal terms in $D^{-1}(s^{\frac{1}{3}}w_j)$ and $D(s^{\frac{1}{3}}w_i)$ never multiply to 1 in the sectors containing C_i and C_j. Hence, in this case W_i and W_j are disjoint.
The only case that remains to be checked is when the natural coordinates on C_i and C_j are the same, when $|i-j|$ is a multiple of six. This happens when the angle between these sectors for the flat metric $|q_0|^{\frac{2}{3}}$ is at least 2π. We can then apply Proposition 6.3 to guarantee that if the highest eigenvalue $e^{s^{\frac{1}{3}}\lambda_i}$ of $D^{-1}(s^{\frac{1}{3}}w_i)$ is in position k_i and the highest eigenvalue $e^{s^{\frac{1}{3}}\lambda_j}$ of $D(s^{\frac{1}{3}}w_j)$ is in position k_j, then the (k_i, k_j)-entry of $U_{i,j}^{-1}$ is not zero. Therefore,

$$d_\omega(h_\infty(w_i), h_\infty(w_j)) \geq \lambda_i + \lambda_j > 0 \ .$$

Again the bound can be made independent of ϵ, hence W_i and W_j are disjoint in this case as well. □

We are then able to describe the global behavior of the harmonic map h_∞.

Corollary 8.11. *Let $\tilde{q}_0$ denote the lift of the cubic differential q_0 to the universal cover $\tilde{\Sigma}$. Let d_∞ be the path distance induced by d_ω on $h_\infty(\tilde{\Sigma}) \subset \mathrm{Cone}_\omega(X)$. Then $h_\infty : (\tilde{\Sigma}, \sqrt{3}\cdot 2^{\frac{1}{6}} d_{\tilde{q}_0}) \to (h_\infty(\tilde{\Sigma}), d_\infty)$ is an isometry.*

Proof. From Equation 8.1 and Lemma 3.4, we know that the induced metric $\hat{g}_s$ rescaled by $s^{-\frac{2}{3}}$ converges pointwise to $3 \cdot 2^{\frac{1}{3}}|\tilde{q}_0|^{\frac{2}{3}}$ and uniformly on every compact set on the complement of the zeros of $\tilde{q}_0$. Let B be a ball centered at a zero of order k as in the setting of Theorem 8.10. Then the induced metric at $h_\infty(0)$ is singular since the total angle is $2\pi + \frac{2k\pi}{3}$. Moreover, in the proof of Theorem 8.10 we showed that radial paths from the origin of q_0-length L are sent to geodesic arcs in the building of length $\sqrt{3} \cdot 2^{\frac{1}{6}} L$. We conclude that $h_\infty(B)$ is isometric to $(B, 3 \cdot 2^{\frac{1}{3}}|q_0|^{\frac{2}{3}})$ and the statement follows. □

Not only is the image of the limiting harmonic map intrinsically a singular flat surface, but $h_\infty(\tilde{\Sigma})$ inherits from the building a $\frac{1}{3}$-translation surface structure as well, which allows us to reconstruct the original cubic differential q_0, up to a positive multiplicative constant.

Corollary 8.12. *The image $h_\infty(\tilde{\Sigma})$ is naturally a $\frac{1}{3}$-translation surface. This structure is induced precisely by the cubic differential $3 \cdot 2^{\frac{1}{3}}\tilde{q}_0$.*

Proof. By Corollary 8.11, we know that $h_\infty(\tilde{\Sigma})$ is a singular flat surface with metric $3 \cdot 2^{\frac{1}{3}}|\tilde{q}_0|^{\frac{2}{3}}$. This defines on $h_\infty(\tilde{\Sigma})$ a holomorphic cubic differential up to multiplication by $e^{i\theta}$. On the other hand, a neighborhood of a regular point p of $h_\infty(\tilde{\Sigma})$ is contained in an apartment A_p by Theorem 8.7 and the directions of the walls of the Weyl-chambers based at p define six special directions. If we identify A_p with

$$\mathbb{A} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\} ,$$

then these directions correspond to the lines $x_i - x_j = 0$ for $i \neq j$. These can be further divided into two groups, defined in accordance with which pair of the triple of coordinates coincide: if we identify the positive Weyl-chamber with

$$\mathfrak{a}^+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 > x_2 > x_3\}$$

the two walls are given by $x_1 - x_2 = 0$ and $x_2 - x_3 = 0$ and the orbits of these lines under the Weyl-group W divide the six walls into two categories, which we call type I and type II. There is only one choice of θ so that along the directions of type II the differential $e^{i\theta}\tilde{q}_0$ is real and positive. □

Remark 8.13. Note that the only scaling factors λ_s that guarantee the existence of the harmonic map h_∞ by Proposition 8.6 are $\lambda_s \asymp s^{\frac{1}{3}}$. Different choices of such permissible λ_s lead only to homothetic asymptotic cones, hence the projective class of the translation surface $h_\infty(\tilde{\Sigma})$ is independent of such a choice of λ_s.

Finally, we relate the geometry of $h_\infty(\tilde{\Sigma})$ with the notion of weak-convexity introduced by Anne Parreau ([Par22]). In brief, she considered the $\mathfrak{a}^+$-valued distance $d_\omega^{\mathfrak{a}^+}$ on $\mathrm{Cone}_\omega(X)$ and defined $d_\omega^{\mathfrak{a}^+}$-geodesics as those paths $\gamma : [0,1] \to \mathrm{Cone}_\omega(X)$ such that for all $t \in [0,1]$

$$d_\omega^{\mathfrak{a}^+}(\gamma(0), \gamma(1)) = d_\omega^{\mathfrak{a}^+}(\gamma(0), \gamma(t)) + d_\omega^{\mathfrak{a}^+}(\gamma(t), \gamma(1)) .$$

Theorem 7.5 shows that the image under h_∞ of a geodesic for the flat metric $|q_0|^{\frac{2}{3}}$ is a geodesic for the distance $d_\omega^{\mathfrak{a}^+}$. We deduce the following:

Corollary 8.14. *The surface $h_\infty(\tilde{\Sigma}) \subset \mathrm{Cone}_\omega(X)$ is weakly convex.*

The argument proves a conjecture of Katzarkov, Noll, Pandit and Simpson ([KNPS17], Conjecture 8.7) in the context of harmonic maps to buildings arising from limits of Hitchin representations in $\mathrm{SL}(3,\mathbb{R})$ along rays of holomorphic cubic differentials. (Formally, this conjecture concerns the "Finsler" distance, obtained by taking a maximum of the vector-valued distance described above, and asks that the image be geodesic in that distance. Here the geodesic nature holds for the vector-valued distance by Theorem 7.5, and such "geodesics" for the two vector-valued distance are clearly geodesics for the Finsler distance. The statement for the Finsler distance then follows.)

Remark 8.15. Note that $h_\infty(\tilde{\Sigma})$ is not totally geodesic in $\mathrm{Cone}_\omega(X)$. One can see this by considering a $\tilde{q}_0$-geodesic arc which passes through a zero of $\tilde{q}_0$ and makes an angle of more than π at that zero. The geodesic connecting the endpoints of that arc lies in an apartment containing the endpoints and is necessarily Euclidean, i.e. has no interior point with an angle between incoming and outgoing directions other than π.

9. Epilogue: Triangle groups

We conclude with an example. Of course, for a closed surface X of high genus, the Labourie-Loftin parametrization of the Hitchin component Hit_3 is given as the cubic differential bundle over the Teichmüller space of X. Thus, even with the results of this paper, an analysis of the limits of this component would require an understanding of the dependence of a diverging sequence of representations on the rays that include them. On the other hand, for triangle groups, we may completely describe the compactification of the $\mathrm{SL}(3,\mathbb{R})$ Hitchin component.

9.1. **The Hitchin component for triangle groups.** Let p, q, r be integers greater than or equal to 2, such that $p^{-1} + q^{-1} + r^{-1} < 1$, and consider the tessellation of the hyperbolic plane by triangles of angles π/p, π/q, π/r. Let $\Gamma_{p,q,r}$, "the oriented (p, q, r)-triangle group", be the group of orientation-preserving isometries of the tiling. Denote the quotient of $\mathbb{H}^2$ by Γ by $\Sigma^{p,q,r}$. Choi-Goldman have shown that the space of convex real projective structures has real dimension 2 for $3 \le p, q < \infty$ and $4 \le r < \infty$ [CG93]. We call these triangle groups *projectively deformable.* The general case of the dimension of the $\mathrm{SL}(n,\mathbb{R})$ Hitchin component for triangle groups was settled by Long-Thistlethwaite [LT19]. Recently, more closely to our point of view, Alessandrini-Lee-Schaffhauser [ALS22] use Higgs bundles to study Hitchin components for orbifolds. In particular, for each projectively deformable (p, q, r) group, the space of cubic differentials has complex dimension one, and we can extend our techniques to give a compactification of the Hitchin component in these cases.

We briefly address cubic differentials for oriented triangle groups.

Lemma 9.1. *Let p, q, r be integers at least 2. The complex dimension of the space of cubic differentials on $\Sigma^{p,q,r}$ is 1 if and only if $p, q, r \geq 3$ and is 0 if any of p, q, r is 2.*

Proof. Consider a local coordinate z with $z = 0$ mapping to an orbifold point of order p. Then $w = z^p$ is a holomorphic coordinate on the orbifold. The condition for a holomorphic cubic differential near $z = 0$ to descend locally to the orbifold is that it can be written as a holomorphic cubic differential in w away from $w = 0$. Thus $z^n\, dz^3$ descends if and only if $n \geq 0$ and $n + 3$ is a multiple of p.

The case $p > 2$ and $n = p - 3$ leads to a pole of order 2 in w, as $z^n\, dz^3 = p^{-3} w^{-2}\, dw^3$. The Riemann surface formed by treating the 3 orbifold points of $\Sigma^{p,q,r}$ as smooth points has genus zero and 3 distinguished points at the triangle vertices of order p, q, r. Thus we seek cubic differentials on $\mathbb{CP}^1$ with poles of order at most 2 at 3 points, thus in a family with one complex parameter. Other values of p, n require lower order poles (or zeros) at the relevant points. There are no nonzero cubic differentials in these cases, in particular when any of p, q, r is 2. ☐

Proposition 9.2. *For $3 \leq p, q < \infty$ and $4 \leq r < \infty$, the real projective structures on $\Sigma^{p,q,r}$ are parametrized by the complex scalings of the nonzero cubic differential q_0.*

The fundamental domain of an oriented triangle orbifold consists of two adjacent triangles, each with vertices at the p, q, r points.

Lemma 9.3. *Let $3 \leq p, q, r < \infty$. A singular Euclidean structure on the orbifold $\Sigma^{p,q,r}$ is induced by a nonzero holomorphic cubic differential if and only if the triangles with vertices projecting to p, q, r are equilateral.*

Proof. Consider the Euclidean structure given by forcing the given triangles to be equilateral, and let z be a flat conformal coordinate on one such triangle. Analytically continue the cubic differential dz^3 to the other triangle of the orbifold. Then we may check the resulting cubic differential is holomorphic on the orbifold, as any monodromy around the orbifold points amounts to translations and rotations by third roots of unity. Such a scalar multiple $\alpha\, dz^3$, for $\alpha \in \mathbb{C}$, is also a holomorphic cubic differential, and together these form the one-dimensional complex vector space of all cubic differentials. ☐

Remark 9.4. On each of these triangle orbifolds $\Sigma^{p,q,r}$, it is useful to choose a particular representative q_0. Give a consistent orientation to the equilateral triangles forming $\Sigma^{p,q,r}$. We choose q_0 so that the sides of the triangles have length 1 and at any given vertex the outgoing edges correspond to the directions on which q_0 is real and positive.

Theorem 9.5. *Consider a family of cubic differentials $se^{i\theta}q_0$ on a closed surface Σ, where $s \to +\infty$ and $\theta \to \theta_\infty$. Let $\rho_{s,\theta}$ be the corresponding family of Hitchin representations. Then the limiting data in terms of singular values and eigenvalues* *converge to those determined in the limit by the ray* $se^{i\theta_\infty}q_0$. *In particular, Theorems A and C hold in this setting.*

Outline of proof. Assume for simplicity that $\theta_\infty = 0$. For a given free homotopy class of loops in Σ, consider the cubic differential $e^{i\theta_\infty}q_0 = q_0$ and the piecewise geodesic path $\tilde{c}^\eta$ considered in Figure 2, modified from a geodesic path along saddle connections in Stokes directions. We compute the holonomy along $\tilde{c}^\eta$ as in Theorems 7.1 and 7.5.

The main consideration is to ensure that we have uniform estimates in θ as $s \to \infty$. Note that the Blaschke metric is independent of θ, and the connection form in (2.2) depends on θ only through the cubic differential q. The upshot is that θ varies in Equation 5.1 and thus in terms of the holonomy along linear paths in Proposition 5.6. It is straightforward to show that the error terms in Proposition 5.6 are uniform in θ as $s \to \infty$, as can be seen in terms of the proof in [Lof07]. In other words, the error term of the form $o(e^{-s^{\frac{1}{3}}\tilde{\mu}^i})$ satisfies

$$\frac{o(e^{-s^{\frac{1}{3}}\tilde{\mu}^i})}{e^{-s^{\frac{1}{3}}\tilde{\mu}^i}} \to 0 \tag{9.1}$$

uniformly in θ as $s \to \infty$.

We also check that the estimates of [DW15] are uniform for θ near θ_∞ as $s \to \infty$. Recall, for the path $\tilde{c}$, in the case in which at least one arc β_i^η has a q_0-Stokes direction as an endpoint, we modify the path to $\tilde{c}^\eta$ to avoid all such endpoints. Thus the same is true in a neighborhood of θ_∞ as $\theta \to \theta_\infty$. There the estimates of [DW15] are uniform in compact sets away from the Stokes directions, in terms of the frames needed in Corollary 4.4 and Theorem 4.6. The product-of-unipotent terms U are also continuous as $\theta \to \theta_\infty$, as they are determined by the geometry of the limiting (in this case, regular) polygon, which varies continuously.

With these estimates in hand, the quantity we consider is the entry-wise L^∞ norm of the holonomy matrix, as given in (7.1). Our first concern is that the largest terms in each matrix match up with positive terms $c_{j,k}$ in the adjacent unipotent matrices in the product in (5.3). This remains true by continuity of the $c_{j,k}$. If c^η contains any saddle connection c_i along a wall of the Weyl chamber, then there is more than one largest eigenvalue of the holonomy along c_i for q_0. This is no longer true if $\theta \neq \theta_\infty$. We must consider the possibility then that this entry-wise L^∞ norm may jump by a positive bounded factor, in the case that $c_{j,k} \to \sum_\alpha c_{j_\alpha,k_\alpha}$ in (7.1). Fortunately the limit we take in Theorem 7.5, in terms of taking a logarithm and dividing by $s^{\frac{1}{3}}$, is insensitive to multiplication by a positive quantity bounded away from 0 and ∞.

Finally, we must ensure that the largest terms in the product of holonomy matrices, upon taking the limit, are not affected by the various error terms we accumulate. This is exactly the uniformity condition (9.1). □

Corollary 9.6. *The previous theorem holds for* Σ *a closed oriented orbifold of hyperbolic type.*

Proof. Consider a smooth finite orbifold cover $\check{\Sigma} \to \Sigma$ and lift the cubic differential to $\check{\Sigma}$. Then each complex scalar multiple of the cubic differential is also invariant under the orbifold deck transformations. □

We next define a compactification of the Hitchin component $\mathrm{Hit}_3(\Delta)$ of representations of a deformable triangle group $\Delta = \Gamma_{p,q,r}$ into $\mathrm{SL}(3,\mathbb{R})$.

First we identify the Hitchin component $\mathrm{Hit}_3(\Gamma_{p,q,r})$ with $\mathbb{C}$ as in Proposition 9.2. We then consider the map

$$\begin{aligned} \mathcal{L}_{\mathfrak{a}} &: \mathbb{C} \to \mathbb{P}(\mathfrak{a}^{\Gamma_{p,q,r}}) \\ se^{i\theta} &\mapsto \{(\log(\sigma_1(\rho_{s,\theta}(\gamma))), \log(\sigma_2(\rho_{s,\theta}(\gamma))), \log(\sigma_3(\rho_{s,\theta}(\gamma))))\}_{\gamma\in\Gamma_{p,q,r}} \end{aligned}$$

where σ_j denotes the j-th largest singular value and $\rho_{s,\theta}$ is the representation corresponding to $se^{i\theta}q_0$. By [Kim04, Theorem B], this map is injective.

Next we also embed S^1 in $\mathbb{P}(\mathfrak{a}^{\Gamma_{p,q,r}})$ as follows. For each $e^{i\theta} \in S^1$, we consider the harmonic map $\hat{h}_\theta : (\tilde{\Sigma}) \to \mathrm{Cone}_\omega(X)$ that results as the ω-limit of the family of harmonic maps $h_s : \tilde{\Sigma} \to X$ parameterized by a ray of cubic differentials $se^{i\theta}q_0$. Then for each such map $\hat{h}_\theta$, we compute the Weyl-chamber lengths of a representative a curve class $[\gamma]$ by considering the image in the principal Weyl chamber $\mathfrak{a}$ of $\hat{h}_\theta([\gamma])$, following the prescription in Corollary 8.12. Naturally this defines a map $S^1 \to \mathbb{P}(\mathfrak{a}^{\Gamma_{p,q,r}})$.

This map $e^{i\theta} \mapsto \mathbb{P}(\mathfrak{a}^{\Gamma_{p,q,r}})$ is also an embedding: to see this, consider the canonical cubic differential q_0 defined in Remark 9.4, and a fixed curve class γ_0 whose q_0-geodesic representative makes an angle of θ_0 with the positive x-axis in a fixed natural coordinate chart. (Here we take $\theta_0 \in [-\frac{\pi}{3}, \frac{\pi}{3}]$ so that the largest entry in the vector is $\cos\theta_0$.) Then for the "rotated" cubic differential $e^{i\theta}q_0$, the largest entry changes to $\cos(\theta_0 + \frac{\theta}{3})$. We regard S^1 as the boundary of the complex plane $\mathbb{C}$ in the usual way, and assert that the usual compactification $\mathbb{C} \cup S^1$ is taken homeomorphically to its image in $\mathbb{P}(\mathfrak{a}^{\Gamma_{p,q,r}})$. This is proved in the next corollary.

Corollary 9.7. *Consider a projectively deformable triangle group. Then the map above* $\mathbb{C} \cup S^1 \to \mathbb{P}(\mathfrak{a}^{\Gamma_{p,q,r}})$ *provides a compactification of the* $\mathrm{SL}(3,\mathbb{R})$ *Hitchin component* $\mathrm{Hit}_3(\Gamma_{p,q,r}) \cong \mathbb{C}$ *of representations of* $\Gamma_{p,q,r}$.

Proof. We need to show that there is a subatlas of boundary charts for the compactification comprising images of a segment T of the circle S^1 and a sector in $\mathbb{C}$ defined by that range T of angles; this amounts to showing that for a family $se^{i\theta}$, as $s \to \infty$ and $\theta \to \theta_\infty$, the images in $\mathbb{P}(\mathfrak{a}^{\Gamma_{p,q,r}})$ of the representations associated to $se^{i\theta}$ converge to those of the harmonic map $\hat{h}_{\theta_\infty}$. This is the content of Theorem 9.5 and that the map $S^1 \to \mathbb{P}(\mathfrak{a}^{\Gamma_{p,q,r}})$ defined on the limiting circle S^1 defined by Corollary 8.12 depended only on the limit of the representations associated to the ray $se^{i\theta_\infty}$. □

A compactification in terms of harmonic maps is more delicate. Proposition 8.6 associates to a family of harmonic maps $h_s : \tilde{\Sigma} \to X$ parameterized by a ray of cubic differentials $se^{i\theta}q_0$ a limiting harmonic map $h_\infty : \tilde{\Sigma} \to \mathrm{Cone}_\omega(X)$, whose image (cf. Corollary 8.12) is a $\frac{1}{3}$-translation surface, isometrically embedded into $\mathrm{Cone}_\omega(X)$.

However, if more general diverging families of harmonic maps are considered, corresponding to families $se^{i\theta_s}q_0$ with $s \to \infty$ and $\theta_s \to \theta_\infty$, their ω-limit $\hat{h}_\infty$ (which exists by the same argument as in Proposition 8.6) are not precluded from depending on the particular family and not only on θ_∞. We may refer to the more classical Teichmüller theory case, where equivariant harmonic maps $u_s : \tilde{\Sigma} \to \mathbb{H}^2$ are parameterized by a family of quadratic differentials Q_s. In that case, if $Q_s = sQ_0$ is a ray, u_s always converges to an equivariant harmonic map $u_\infty : \tilde{\Sigma} \to T$ to a real tree given by projection onto the leaves of the vertical foliation of Q_0 ([Wol95]). What is independent of the family is the projective class of the vertical foliation of the Hopf differential of the limiting harmonic map. In our setting, the harmonic map approach identifies the boundary points of a compactification of the Hitchin component for $\Gamma_{p,q,r}$ with projective classes of $\frac{1}{3}$-translation surfaces.

(The Euclidean $(3,3,3)$ triangle group also can be studied from this point of view. There is no hyperbolic structure on this orbifold, but there is still a nowhere-vanishing cubic differential, which explicitly leads to the Ţiţeica example on the orbifold universal cover. As the cubic differential scales to infinity, the Ţiţeica scales as well, and the limiting harmonic map into the real building simply covers a single apartment.)

Remark 9.8. We conclude with an informal remark. In the setting of these triangle groups, the limiting harmonic maps to buildings take on enough of a combinatorial nature that we may display how some of the constructions in Section 8 apply.

The simplest triangle group for this is the $(3,3,4)$ group. One can visualize the action of this group on the hyperbolic plane in terms of a triangulation: two of the vertices of each triangle are vertices with a star of six triangles, and the remaining, say special, vertex is a vertex with a star of eight triangles. Distinct special vertices of adjacent triangles share the same opposite edge.

For the cubic differential q_0 defined in Remark 9.4, the harmonic map takes each triangle to an equilateral triangle in a building. In terms of the local geometry of the map, the six triangles in the image of the star around a non-special vertex will lie in a common apartment in the building, but the eight triangles in the image around a special vertex cannot, due for example to cone angle considerations. We return to this local geometry momentarily.

More globally, we comment a bit on some apartments which meet the image of the harmonic map. The reader may refer to Figure 4 as support. Note that a geodesic segment between (images of) special vertices in adjacent triangles – this segment will meet the opposite side orthogonally – may be extended to an infinite geodesic which subtends one of the angles, choose it always to be on the right, at each special vertex of exactly π. This infinite (oriented) geodesic is the boundary of a Euclidean strip of parallel geodesics in the image of the harmonic map whose preimages limits on a pair of distinct endpoints. Each of these strips embed isometrically in an apartment which meets the image of the harmonic map in a region that contains the strips. Since there is a geodesic segment between images of special vertices that bisects each

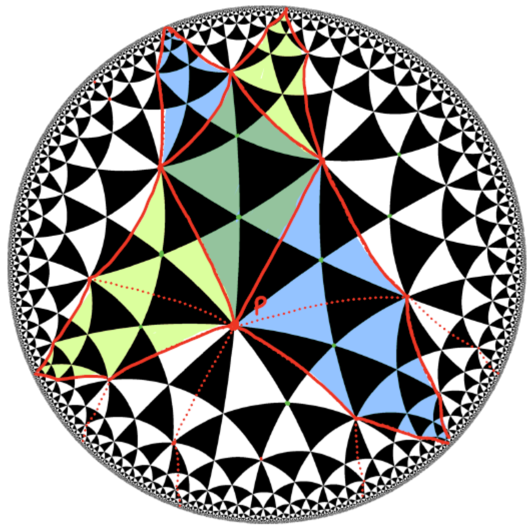

FIGURE 4. Tessellation of the hyperbolic plane by fundamental domains for the $(3,3,4)$ triangle group. The harmonic map sends each hyperbolic triangle to a Euclidean triangle in an apartment of X. Two highlighted strips have images which each lie in a single apartment. Their intersection is a rhombus connecting special vertices with a star of eight triangles.

triangle in the star of the image of a special vertex, a neighborhood of the image of a special vertex is covered by eight such strips, and two (non-disjoint) strips meet in a rhombus (angles alternately $\frac{\pi}{6}$ and $\frac{\pi}{3}$) whose vertices are images of special vertices.

Note that each such strip meets four of the triangles in the star around the image of a special vertex so that the apartment containing this strip meets those four triangles, i.e. those four adjacent triangles around the image of a special vertex are in an embedded flat in the building. On the other hand, five adjacent triangles around that image vertex cannot be in an embedded flat (nor apartment) since that flat would then force there to be a sixth triangle with one edge shared with the terminal triangle and one shared with the initial triangle. That sixth triangle would combine with the three remaining image triangles in the image to form an embedded cone in the building of cone angle $\frac{4\pi}{3}$, which cannot exist in an NPC simplicial complex (e.g. points equidistant but on opposite sides of the cone point are joined by distinct geodesics on opposite sides of the cone point).

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